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On the E -polynomials of a family of Character Varieties

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On the E -polynomials of a family of Character Varieties

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We compute the E -polynomials of a family of twisted character varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ by proving they have polynomial count, and applying a result of N. Katz on the counting functions.

To compute the number of \mathbb{F}_q -points of these varieties as a function of q , we used a formula of Frobenius. Our calculations made use of the character tables of $\mathrm{Gl}_n(q)$ and $\mathrm{Sl}_n(q)$, previously computed by J. A. Green and G. Lehrer, and a result of Hanlon on the Möbius function of a subposet of set-partitions.

The Euler Characteristics of the $\mathcal{M}^g(\mathrm{Sl}_n)$ are calculated then with these polynomial.

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List of Notation

| | | |
|--|---|----|
| ζ_n | a primitive n th-root of unity | 1 |
| $[y, x]$ | the commutator of x and y , $xyx^{-1}x^{-1}$ | 1 |
| $\mathcal{P}, \mathcal{P}_n$ | sets of partitions, and partitions of size n | 7 |
| $\mathcal{P}^\Xi, \mathcal{P}_n^\Xi$ | sets of Ξ -partitions, multi-partitions | 9 |
| Λ' | conjugate multipartition of Λ' | 9 |
| $\square, h(\square)$ | a box in a Ferrers diagram, its hook-length | 8 |
| $ \Lambda $ | size of the multipartition Λ | 9 |
| $H_\lambda, H_\Lambda, H_\tau$ | Hook polynomials | 10 |
| $\mathcal{H}_\lambda, \mathcal{H}_\Lambda, \mathcal{H}_\tau$ | Normalized hook polynomials | 11 |
| $(\tau)_{d,\lambda}$ | multiplicity of (d, λ) in τ | 10 |
| $m_{d,\lambda}, m_{d,\lambda}(\Lambda)$ | multiplicity of (d, λ) in Λ | 10 |
| $\tau/(t_d, t_m)$ | quotient type of τ by t_d and t_m | 63 |
| $A \hookrightarrow B$ | injective | 23 |
| $A \twoheadrightarrow B$ | surjective | 23 |
| $x \mapsto y$ | x is mapped to y | 12 |
| $A _K$ | restriction of A to K | 24 |
| ${}^t N$ | transpose of N | 23 |
| $\overrightarrow{\lambda}$ | a list $(\lambda_1, \lambda_2, \dots, \lambda_m)$ | 59 |
| $\mu(,)$ | Möbius function of a poset | 13 |
| Π_n | set-partitions of $\{1, 2, \dots, n\}$ | 13 |

| | | |
|--------------------------|---|----|
| Π_n^ρ, Π^ρ | set-partitions of fixed by ρ | 14 |
| $\langle \delta \rangle$ | Group generated by δ | 50 |
| \widehat{G} | dual, or character group, of G , $\text{Hom}(G, \mathbb{C}^\times)$ | 16 |
| $\text{Irr}(G)$ | set of irreducible characters or representatinos of G | 15 |
| $Frob_q$ | Frobenius automorphism $x \mapsto x^q$ | 23 |
| $G \curvearrowright X$ | G acts on X | 55 |
| X/G | set of G -orbits of X | 22 |
| X^ρ | ρ -fixed points of X | 23 |
| $Stab(x)$ | stabilizer of x | 22 |
| Gx | G -orbit of x | 56 |
| $[x]$ | orbit of x | 21 |
| $\{\gamma\}$ | $Frob_q$ -orbit of γ | 25 |
| $\deg(\alpha)$ | degree of α , the size $ \{\alpha\} $ of its $Frob_q$ -orbit | 24 |
| $\deg^t(\alpha)$ | newdegree of α relative to t , $\deg([\alpha])$ for $[\alpha] \in \Gamma/\langle \delta^s \rangle$ | 55 |
| $(G)_d$ | elements of G whose degree divides d , $G^{Frob_q^d}$ | 83 |
| Σ_g | a genus g Riemann Surface | 21 |
| $X//G$ | Geometric Invariant Theory quotient of X by G | 1 |
| $h^{p,q;j}, h_c^{p,q;j}$ | Hodge numbers | 32 |
| $H(x, y, t; X)$ | Hodge polynomial | 32 |
| $H_c(x, y, t; X)$ | compactly supported Hodge polynomial | 32 |
| $E(x, y; X)$ | E -polynomial, $H_c(x, y, -1; X)$ | 32 |
| $E(q; X)$ | E -polynomial, $E(\sqrt{q}, \sqrt{q}; X)$ | 33 |
| Γ | Colimit of character groups of \mathbb{F}_q^\times | 23 |

| | | |
|---|---|----|
| Θ | Frobenius orbits of $\Gamma, \Gamma / \text{Frob}_q$ | 25 |
| $\sigma_\alpha, \tau_\alpha$ | $\widehat{\mathbb{F}}_q$ -action by α | 27 |
| $\mathcal{M}^g(\text{Sl}_n)$ | Twisted Character Variety for Sl_n | 1 |
| $\mathcal{M}^g(\text{Sl}_n)(q)$ | Twisted Character Variety for $\text{Sl}_n(q), \mathcal{M}^g(\text{Sl}_n(q))$ | 2 |
| $N_n^g(q)$ | Number of \mathbb{F}_q -points of $\mathcal{M}^g(\text{Sl}_n)$ | 1 |
| $\tilde{N}_n^g(q)$ | Number of solutions of $\prod_{i=1}^g [A_i, B_i] = \zeta_n Id$ | 43 |
| $f(q) _{q=1}$ | The evaluation of f at $q = 1$ | 48 |
| $\sum_{\substack{i,j \\ c(i,j)}} f(i, j)$ | Sum of f over (i, j) satisfying condition $c(i, j)$ | 3 |

Chapter 1

Introduction

The aim of this work is to compute the E -polynomial of a family of twisted character varieties for the group Sl_n . This is accomplished by counting the number $N_n^g(q)$ of points that these varieties have over finite fields, following similar ideas as those from [9].

The Character Varieties we study are the affine GIT quotients

$$\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{C})) := \{A_1, B_1, \dots, A_g, B_g \in \mathrm{Sl}_n(\mathbb{C}) \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n I_n\} // \mathrm{PGL}_n(\mathbb{C}) \quad (1.1)$$

where $[A, B] = ABA^{-1}B^{-1}$, ζ_n is a primitive n -th root of 1, and the group $\mathrm{PGL}_n(\mathbb{C})$ acting by conjugation. They appeared naturally in [9], (2.2.13) to show that the PGL_n -Character Varieties have an orbifold structure.

The E -polynomial of a variety X is

$$E(q; X) := H_c(\sqrt{q}, \sqrt{q}, -1; X) \quad (1.2)$$

where

$$H_c(x, y, t; X) := \sum h_c^{p,q;j}(X) x^p y^q t^j, \quad (1.3)$$

with the $h^{p,q;j}$ being the Mixed Hodge numbers of X , defined by Deligne in [2], [3].

The strategy to compute the E -polynomials of $\mathcal{M}^g(\mathrm{Sl}_n)$ is to prove that such varieties have polynomial count, which means that there is a polynomial $E_n(q) \in \mathbb{Z}[q]$ such that $\#\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q)) = E_n(q)$ for sufficiently many prime powers q , in the sense described in 3.3.

According to Katz's result from the Appendix of [9], the counting polynomials E_n must coincide with the E -polynomial of the variety.

To compute $\#\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q))$ we regard its \mathbb{F}_q -points as:

$$\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q)) = \{A_1, B_1, \dots, A_g, B_g \in \mathrm{Sl}_n(\mathbb{F}_q) \mid [A_1, B_1] \dots [A_g, B_g] = \zeta_n I_n\} // \mathrm{PGL}_n(\mathbb{F}_q) \quad (1.4)$$

assuming $n|q-1$, a necessary condition for a primitive n th root of unity ζ_n in \mathbb{F}_q to exists.

The problem is then reduced (Corollary 3.1.3, and Remark 3.4.3) to finding an expression for $N_n^g(q)$, defined as the number $\tilde{N}_n^g(q)$ of solutions of the equation

$$[A_1, B_1] \dots [A_g, B_g] = \zeta_n I_n, \quad A_i, B_i \in \mathrm{Sl}_n(\mathbb{F}_q) \quad (1.5)$$

divided by $|\mathrm{PGL}_n(\mathbb{F}_q)|$.

The number of solutions of an equation like (1.5) over the finite group $G = \mathrm{Sl}_n(\mathbb{F}_q)$ can be calculated thanks to the following formula of Frobenius

$$\sum_{\chi \in \mathrm{Irr}(G)} \left(\frac{|G|}{\chi(1)} \right)^{2g-1} \chi(z) \quad (1.6)$$

where the sum is taken over all irreducible characters χ of G (Proposition 2.2.1).

With this end in mind, we need the character table of $\mathrm{Sl}_n(\mathbb{F}_q)$, which was computed in [12] and described in [11] by a technique known as Clifford Theory, as components of restrictions of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$. These were already calculated in [7].

Thanks to these character tables, formula (1.6) and Katz's Theorem, we prove the following:

Theorem 1.0.1. *The Character Varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ have polynomial count and their E -polynomials are given by:*

$$E(q; \mathcal{M}^g(\mathrm{Sl}_n)) = \sum_{\tau, t} \left(q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} t^{2g-1} C_{\tau}^t. \quad (1.7)$$

Remark 1.0.1. The sum is taken over types τ of size n (defined in 2.1.9) and divisors t of n . The polynomials $q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{q-1}$ are described in 2.2.20 and the coefficients C_{τ}^t are defined in (5.6) and computed in (5.24).

Manipulating the terms in this summation we get the following

Corollary 1.0.2.

$$E(q; \mathcal{M}^g(\mathrm{Sl}_n)) = \frac{1}{(q-1)^{2g-2}} \sum_{\substack{|\tau|=n \\ \tau/(t_d, t_m) = \widehat{\tau} \\ (\widehat{d}; t_d t_m) = 1}} \mu(t_d) \frac{O^{2g}(t_d t_m)}{t_d t_m} \left(q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q) \right)^{2g-2} C_{\widehat{\tau}}^0. \quad (1.8)$$

Remark 1.0.2. In last formula, $O^{2g}(t) = \sum_{d|t} \mu(t/d) d^{2g}$ is the number of elements in $(\mathbb{C}^\times)^{2g}$ of order t , the coefficients C_τ^0 are $C_\tau/(q-1)$ for the C_τ computed in [9] (see (5.22) for a formula), t_d and t_m are divisors of n and the sum is taken over a well described set of tuples $(\tau, \widehat{\tau}, t_d, t_m)$ (see 5.26 for details).

Remark 1.0.3. The polynomials found in 5.26 give the correct number of points $N_n^g(q)$ for those q satisfying

$$q \equiv \begin{cases} 1 \pmod{n} & \text{when } n \text{ is odd} \\ 1 \pmod{2n} & \text{when } n \text{ is even.} \end{cases} \quad (1.9)$$

Such stronger condition on q for n even is necessary. For instance, for $n = 2$ the counting function $N_2^g(q)$ is a quasi-polynomial for those $q \equiv 1 \pmod{n}$ of period at most 2, as we will see in Chapter 4.

From these results, and the equidimensionality of the $\mathcal{M}^g(\mathrm{Sl}_n)$ we get the following:

Corollary 1.0.3. *The E -polynomial of $\mathcal{M}^g(\mathrm{Sl}_n)$ is palindromic and monic. In particular, the Character Varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ are connected.*

As pointed out in 3.3.3, one can compute the Euler characteristic of the $\mathcal{M}^g(\mathrm{Sl}_n)$ by evaluating their E -polynomials at $q = 1$, getting

Corollary 1.0.4. *The Euler Characteristic of the Character Varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ is 1 for $g = 1$ and $\mu(n)n^{4g-3}$ for $g \geq 2$.*

Remark 1.0.4. The degree of the E -polynomial of $\mathcal{M}^g(\mathrm{Sl}_n)$ is $2(g-1)(n^2-1)$. Since the character variety is equidimensional $2(g-1)(n^2-1)$ must thus be its dimension. This is consistent with [9],(2.2.13)

Remark 1.0.5. The results of Corollaries 1.0.4 and ?? are consistent with those in [9], more precisely with Corollary 1.1.1 on the Euler Characteristic of the PGL_n -Character Varieties and the claims along the proof of Theorem 2.2.12 on the orbifold structure

$$\mathcal{M}^g(\mathrm{Sl}_n) \twoheadrightarrow \mathcal{M}^g(\mathrm{PGL}_n).$$

The dissertation is organized as follows:

In Chapter 2 we go over the basics of Combinatorics and Representation Theory that is going to be needed. In Chapter 3 we define the Character Varieties we want to study and their E -polynomials. In Chapter 4 we elaborate upon the case $n = 2$ to show how irreducible characters of $\mathrm{GL}_2(q)$ split and to illustrate the way calculations with formula (1.6) carry over. In Chapter 5 we outline the computation of the number of \mathbb{F}_q -points of $\mathcal{M}^g(\mathrm{Sl}_n)$ by applying the Möbius Inversion Formula several many times. In Chapter 6 we compute in detail some claims that were left unproven, finishing with the calculation of $N_n^g(q)$ and hence that of $E(q; \mathcal{M}^g(\mathrm{Sl}_n))$.

Chapter 2

Preliminaries

We are going to need some definitions and basic results from Combinatorics and Representation Theory. For definitions and notation on partitions and multi-partitions we follow [18]. The Frobenius formula for counting solutions of equations on finite groups is explained in [15]. For the basics on Clifford Theory we follow [11].

2.1 Combinatorics

2.1.1 Partitions

Notation. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ we have $\lambda_1 \geq \dots \geq \lambda_l \geq 1$ and write $l(\lambda)$ for its length l , and $|\lambda|$ for its size $\sum \lambda_i$.

Let \mathcal{P}_n note the set of partitions of n and

$$\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}_n.$$

We consider the empty partition \emptyset as having size and length 0, so that \mathcal{P}_0 has one element.

Remark 2.1.1. We usually regard $\lambda_i = 0$ whenever $i > l(\lambda)$ for the ease of notation in formulae.

Definition 2.1.1. For a partition λ we note λ' its conjugate partition $(\lambda'_1, \dots, \lambda'_{l'})$, where λ'_i is the number of parts of λ not smaller than i . In particular, $l' = l(\lambda') = \lambda_1$ and $l = l(\lambda) = \lambda'_1$.

Remark 2.1.2. The conjugate partition λ' is usually thought as the one whose Ferrers' diagram (1.3 of [17], for instance) is obtained by flipping that of λ across its main diagonal.

Definition 2.1.2. We note $\square \in \lambda$ a box in the aforementioned diagram, and if \square is in position (i, j) then its hook length $h(\square)$ in λ is defined as

$$h(\square) := \lambda_i + \lambda'_j - i - j + 1.$$

Definition 2.1.3. There is an inner product for partition given by

$$\langle \lambda, \nu \rangle := \sum_{j \geq 1} \lambda'_j \nu'_j$$

where $\lambda, \nu \in \mathcal{P}$, and the number

$$n(\lambda) := \frac{1}{2}(\langle \lambda, \lambda \rangle - |\lambda|) = \sum_{i \geq 1} (i-1)\lambda_i.$$

Example: 2.1.3. For the partition $\lambda = (7, 5, 5, 3, 1)$, we have $\lambda' = (5, 4, 4, 3, 3, 1, 1)$, $|\lambda| = |\lambda'| = 21$ and the Ferrers diagram will be:

| $\lambda_i \backslash \lambda'_i$ | 5 | 4 | 4 | 3 | 3 | 1 | 1 |
|-----------------------------------|-----------|----------------|----------------|----------------|----------------|-----------|-----------|
| 7 | \square | \square | \square | \square | \square | \square | \square |
| 5 | \square | \square | \square | \square | \square | | |
| 5 | \square | \blacksquare | \blacksquare | \blacksquare | \blacksquare | | |
| 3 | \square | \blacksquare | \square | | | | |
| 1 | \square | | | | | | |

the black squares corresponds to the hook of $\square_{3,2}$ the box in position $(3, 2)$ and its hook length is:

$$h(\square_{3,2}) = 5 + 4 - 3 - 2 + 1 = 5.$$

2.1.2 Multi-Partitions

We will consider collections Ξ of finite sets $X \in \Xi$. In Section 2.2.4 they will be $Frob_q$ -orbits of characters.

Definition 2.1.4. For such a collection Ξ , we define an Ξ -partition (or multi-partition) as a function $\Lambda : \Xi \rightarrow \mathcal{P}$.

The size of a multi-partition Λ being defined as

$$|\Lambda| := \sum_{X \in \Xi} |X| |\Lambda(X)|.$$

Let \mathcal{P}_n^Ξ note the set of multi-partitions of size n and

$$\mathcal{P}^\Xi := \bigcup_{n \geq 0} \mathcal{P}_n^\Xi.$$

Definition 2.1.5. For $\Lambda \in \mathcal{P}_n^\Xi$ we define its conjugate $\Lambda' \in \mathcal{P}_n^\Xi$ by

$$\Lambda'(X) := \Lambda(X)' \in \mathcal{P}.$$

Definition 2.1.6. Similarly we extend the notion of number for multi-partitions as

$$n(\Lambda) := \sum_{X \in \Xi} |X| n(\Lambda(X)).$$

Notation. We will usually regard $\Lambda \in \mathcal{P}^\Xi$ as a function from $\bigcup_{X \in \Xi} X$ to \mathcal{P} constant on the sets X . Thus, for $x \in X$, $\Lambda(x) := \Lambda(X)$.

Definition 2.1.7. The support of Λ is the union of all those $X \in \Xi$ mapped to a nonzero partition $\Lambda(X) \neq \emptyset$.

Definition 2.1.8. Given an integer $d \geq 1$ and a partition $\lambda \in \mathcal{P}$ we define the multiplicity of (d, λ) in Λ as

$$m_{d,\lambda} = m_{d,\lambda}(\Lambda) := \# \{X \in \Xi \mid |X| = d, \Lambda(X) = \lambda\},$$

when $\lambda \neq 0$, and set $m_{d,0} = 0$ as a convention.

Definition 2.1.9. Let $\tau(\Lambda)$ be the collection of multiplicities $(m_{d,\lambda})_{d \geq 1, \lambda \in \mathcal{P}}$ and call it the type of Λ . Its size is given by $|\tau| := |\Lambda| = \sum m_{d,\lambda} d |\lambda|$.

Notation. We write $(\tau)_{d,\lambda}$ or $m_{d,\lambda}(\tau)$ for the multiplicity $m_{d,\lambda}(\Lambda)$ of any multi-partition Λ of type τ .

Definition 2.1.10. The support of τ is the set of pairs (d, λ) with a nonzero associated multiplicity $(\tau)_{d,\lambda}$.

Definition 2.1.11. We write $\square \in \Lambda$ for a box in the Ferrers diagram of one of the partitions $\Lambda(X)$ and define its hook length $h(\square)$ in Λ as $|X|$ times its hook length in $\Lambda(X)$.

2.1.3 Hook Polynomials

Definition 2.1.12. Given a partition λ we associate to it its hook polynomial

$$H_\lambda(q) := \prod_{\square \in \lambda} (q^{h(\square)} - 1). \quad (2.1)$$

In a similar fashion, for a multi-partition Λ we define:

$$H_\Lambda(q) := \prod_{\square \in \Lambda} (q^{h(\square)} - 1) = \prod_{X \in \Xi} H_{\Lambda(X)}(q^{|X|}). \quad (2.2)$$

Notation. Since the $H_\Lambda(q)$ only depends on the type τ of Λ we will write H_τ for H_Λ .

Most formulas will look more transparent if we introduce the normalized version of the hook polynomial.

Definition 2.1.13. For $\lambda \in \mathcal{P}$ and $\Lambda \in \mathcal{P}_n(\Gamma)$ any map of type $\tau = \{m_{d,\lambda}\}$, the corresponding normalized hook polynomials are:

$$\mathcal{H}_\lambda(q) := q^{-\frac{1}{2}\langle \lambda, \lambda \rangle} \prod_{\square \in \lambda} (1 - q^{h(\square)}) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \quad (2.3)$$

and

$$\mathcal{H}_\Lambda(q) := \prod_{\{\gamma\}} \mathcal{H}_{\Lambda(\gamma)}(q^{deg(\gamma)}) = \prod_{d,\lambda} \mathcal{H}_\lambda(q^d)^{m_{d,\lambda}}. \quad (2.4)$$

Since \mathcal{H}_Λ depends only on the type τ of Λ , we also write $\mathcal{H}_\tau(q) = \mathcal{H}_\Lambda(q)$.

Remark 2.1.4. By the symmetry of the factors of (2.3) we have the identity

$$\mathcal{H}_\lambda(q^{-1}) = (-1)^{|\lambda|} \mathcal{H}_{\lambda'}(q). \quad (2.5)$$

Definition 2.1.14. For a type τ the size, number, length and hook polynomial of τ are then defined as those of Λ for any Λ of type τ . In a similar way, we can also define the conjugate τ' as that of Λ' .

2.1.4 Möbius Inversion formula:

Let (P, \geq) be a finite poset, i.e. a finite partially ordered set.

Definition 2.1.15. To every function $f : P \rightarrow \mathbb{C}$ we assign

$$\widehat{f} : P \rightarrow \mathbb{C}$$

$$\widehat{f}(a) := \sum_{b \geq a} f(b)$$

the accumulated sums of f with respect to P .

Remark 2.1.5. We see that $f \mapsto \widehat{f}$ is a linear operator in $\text{Hom}_{\mathbb{C}}(\mathbb{C}^P, \mathbb{C}^P)$ whose matrix M_{Σ} is given by

$$(M_{\Sigma})_{a,b} = \begin{cases} 1 & \text{if } b \geq a \\ 0 & \text{otherwise.} \end{cases}$$

the adjacency matrix of (the digraph induced by) P .

Since P is a poset, this matrix will be lower triangular with respect to some total ordering refining that of P , and will have only 1's in its main diagonal.

Therefore its inverse $\mu \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^P, \mathbb{C}^P)$ has also only 1's in the diagonal and integer entries, for writing M_{Σ} as $Id + A$, with A strictly lower triangular (hence nilpotent), the alternating sum

$$Id - A + A^2 - A^3 \dots$$

for $(Id + A)^{-1}$ results finite.

Definition 2.1.16. The function $\mu(a, b)$ given by the entries $(\mu)_{a,b}$ of the matrix μ is called the Möbius Function for P .

Remark 2.1.6. By definition of μ we have the Möbius Inversion Formula:

$$f(a) := \sum_{b \geq a} \mu(a, b) \widehat{f}(b). \quad (2.6)$$

Example: 2.1.7. (Divisors) let P be the set of positive divisors of a fixed n with the ordering given by reversed divisibility, more precisely $a \geq b$ if and only if $a|b$.

The accumulated sums for f are $\widehat{f}(m) := \sum_{d|m} f(d)$ and the inversion formula for this case is the well known

$$f(m) := \sum_{d|m} \widehat{f}(d) \mu(m/d) \quad (2.7)$$

where

$$\mu(m) := \begin{cases} (-1)^{\#\{\text{primes dividing } m\}} & \text{for square-free } m, \\ 0 & \text{otherwise.} \end{cases}$$

In other words $\mu(a, b) = \mu(a/b)$ if b divides a and 0 if not.

Example: 2.1.8. (Set-partitions) Let us take P as the collection Π_n of set-partitions of the set $\{1, 2, \dots, n\}$ for a fixed n and the order being given by reversed refinement.

For instance, in Π_4 we have

$$1234 \geq 12|34 \geq 1|2|34 \geq 1|2|3|4.$$

Then the Möbius function is:

$$\mu(\nu, \pi) = (-1)^{s-r} \prod_{i=1}^s (i-1)!^{r_i}$$

where s, r are the numbers of parts of ν and π (respectively), r_i is the number of parts of π containing exactly i parts of ν .

Example: 2.1.9. (Fixed set-partitions) Given a poset Π_n as in last example, and a $\rho \in \mathbb{S}_n$ we take the subposet Π_n^ρ of those set-partitions fixed by ρ . In general, its Möbius function is not easy to compute. In [8] Hanlon computed $\mu(\widehat{0}, \widehat{1})$ where $\widehat{0}$ and $\widehat{1}$ stands for the bottom and the top element in such a poset, namely those partitions with n and 1 parts, respectively. His results reads:

$$\mu(\widehat{0}, \widehat{1}) = \begin{cases} \mu(d)(-d)^{m_d-1}(m_d-1)! & \text{if } \rho \text{ is a product of } m_d \text{ } d\text{-cycles,} \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

Remark 2.1.10. Thhe last example played its role in the computation of the E -polynomials of the twisted Gl_n -character varieties in [9]. We are going to use it for the Sl_n -case as well.

Remark 2.1.11. Example 2.1.7 is also going to be important for many computations, as we will see in Chapter 5 and Chapter 6.

As an application of it, let us compute the number $O^N(n)$ of elements of a order in the torus $(\mathbb{C}^\times)^N$. Writing $\widehat{O}^N(m)$ for the number of elements in the m -torsion subgroup, the following two observations leads to the desired formula:

$$\bullet \quad \widehat{O}^N(m) = \sum_{d|m} O^N(d).$$

- $\widehat{O}^N(m) = m^N$.

Therefore

$$O^N(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d^N. \quad (2.9)$$

2.2 Representation Theory

In this section we will list the facts from Representation Theory we will need. We assume G is a finite group, and our representations will all have finite dimension. For proofs and details we refer the reader to [14], [6].

A representation of G is a finite dimensional \mathbb{C} -vector space V together with a group homomorphism

$$\rho : G \rightarrow \text{Aut}_{\mathbb{C}}(V).$$

Its character $\chi = \chi_{\rho}$ is the class function obtained from composition with the trace

$$\chi_{\rho}(g) := \text{tr}(\rho(g)).$$

Since the character χ_{ρ} of a representation ρ determines it uniquely up to isomorphism, a representation is usually identified with its character.

A representation is said to be irreducible if it does not have nontrivial invariant subspaces. In such case, its character is also called irreducible.

Notation. For a finite group G we note $\text{Irr}(G)$ the set of its irreducible characters. Whenever it is clear from the context, it will also represent the set of (isomorphism classes of) irreducible representations of G .

In an irreducible representation (V, ρ) , Schur's Lemma says that all the endomorphisms $\text{Hom}_\rho(V, V)$ are the scalar multiplications $\zeta Id : x \mapsto \zeta x$ with $\zeta \in \mathbb{C}$.

There is a natural inner product $\langle \cdot, \cdot \rangle$ for characters given by

$$\langle \chi, \chi' \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)}.$$

We also recall the Orthogonality relations:

For $\chi, \chi' \in \text{Irr}(G)$ then

$$\langle \chi, \chi' \rangle = \begin{cases} 1 & \text{if } \chi = \chi' \\ 0 & \text{otherwise.} \end{cases} \quad (2.10)$$

For $g, h \in G$

$$\sum_{\chi \in \text{Irr}(G)} \chi(g) \overline{\chi'(h)} = \begin{cases} |C_g(G)| & \text{if } g \text{ and } h \text{ are conjugate} \\ 0 & \text{otherwise} \end{cases} \quad (2.11)$$

where $C_g(G)$ is the centralizer of g in G (2.2 in [6], for instance).

2.2.1 Character Groups

Definition 2.2.1. For G a finite group, we define its character group (or dual) as $\widehat{G} := \text{Hom}(G, \mathbb{C}^\times)$.

Remark 2.2.1. It has a structure of abelian group given by pointwise multiplication. Its identity element is the trivial (or principal) character $1_G : g \mapsto 1$.

Remark 2.2.2. For G abelian $\widehat{G} = \text{Irr}(G)$.

Remark 2.2.3. When G is finite and cyclic, \widehat{G} is easily seen to be cyclic of same order, then $G \cong \widehat{\widehat{G}}$ in a non canonical way, and the same holds for any finite abelian group G .

Remark 2.2.4. The canonical map $G \longrightarrow \widehat{\widehat{G}}$ is an isomorphism.

Remark 2.2.5. Since \mathbb{C}^\times is divisible, any short exact sequence of abelian groups

$$0 \longrightarrow K \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow 0$$

induces its dual

$$0 \longrightarrow \widehat{Q} \xrightarrow{\pi^*} \widehat{G} \xrightarrow{i^*} \widehat{K} \longrightarrow 0$$

leading us to the conclusion

- The dual of a quotient is a subgroup of the dual: namely, the dual of $Q = G/K$ is the subgroup of those $\phi \in \widehat{G}$ that vanish over K .
- The dual of a subgroup is a quotient of the dual: namely, the dual of $K \subseteq G$ is the quotient of \widehat{G} by the dual \widehat{Q} .

Remark 2.2.6. For G cyclic there is only one subgroup and only one quotient of order d for every d divisor of $|G|$. Given $H \subseteq G$ and $g \in H$, $g \in H$ if and only if $|g|$ divides $|H|$.

Remark 2.2.7. For G abelian,

$$\sum_{\phi \in \widehat{G}} \phi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

from (2.11), for instance.

Remark 2.2.8. Given a subgroup \widehat{Q} of \widehat{G} , by Remarks 2.2.7 and 2.2.5 we have

$$\sum_{\phi \in \widehat{Q}} \phi(g) = \begin{cases} |Q| & \text{if } g \in K \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2.9. For cyclic G , by Remarks 2.2.6 and 2.2.8, the computation of a character sum over a subgroup of \widehat{G} is reduced to a question of divisibility of integers. This is going to be important in the proof of Lemma 6.3.3.

2.2.2 Counting solutions of equations in finite groups

Let G be a finite group and A some abelian group like \mathbb{C} or $\mathbb{C}^{n \times n}$.

Definition 2.2.2. For a function $f : G \rightarrow A$ we note

$$\int_G f(x) dx := \frac{1}{|G|} \sum_{x \in G} f(x)$$

the integral of f with respect to the Haar Measure of G .

Proposition 2.2.1. *Given $z \in G$, the number of $2g$ -tuples $(x_1, y_1, \dots, x_g, y_g)$ satisfying $[y_1, x_1] \dots [y_g, x_g]z = 1$ is:*

$$\#\{[y_1, x_1] \dots [y_g, x_g]z = 1\} = \sum_{\chi \in \text{Irr}(G)} \chi(z) \left(\frac{|G|}{\chi(1)} \right)^{2g-1}. \quad (2.12)$$

Proof. Given $\rho : G \rightarrow \text{Aut}(V)$ an irreducible representation of G and χ its character, let us consider for a fixed $x \in G$ the average

$$\int_G \rho(yxy^{-1}) dy. \quad (2.13)$$

Since it commutes with the G -action, it must be, by Schur's lemma, a scalar map ζId .

By taking traces we get:

$$\begin{aligned}
\int_G \rho(yxy^{-1})dy &= \zeta Id \\
\int_G \text{tr}(\rho(yxy^{-1}))dy &= \zeta \text{tr}(Id) \\
\int_G \chi(yxy^{-1})dy &= \zeta \chi(1) \\
\int_G \chi(x)dy &= \zeta \chi(1) \\
\frac{\chi(x)}{\chi(1)} &= \zeta
\end{aligned} \tag{2.14}$$

thus, the average (2.13) becomes

$$\int_G \rho(yxy^{-1})dy = \frac{\chi(x)}{\chi(1)} Id. \tag{2.15}$$

Multiplying by $\rho(x^{-1})$ from the right we get:

$$\int_G \rho(yxy^{-1}x^{-1})dy = \frac{\chi(x)}{\chi(1)} \rho(x)^{-1}$$

summing over all $x \in G$ and dividing by $|G|$ again we end up with

$$\iint_{G^2} \rho([y, x])dxdy = \int_G \frac{\chi(x)}{\chi(1)} \rho(x)^{-1} dx. \tag{2.16}$$

Since the left hand side of this equation is invariant under G -conjugation, it is also a scalar transformation. Taking traces again we conclude

$$\begin{aligned}
\iint_{G^2} \rho(yxy^{-1}x^{-1})dxdy &= \int_G \frac{\chi(x)\chi(x^{-1})}{\chi(1)^2} dx Id \\
&= \int_G \frac{\chi(x)\overline{\chi(x)}}{\chi(1)^2} dx Id \\
&= \frac{1}{\chi(1)^2} Id.
\end{aligned} \tag{2.17}$$

Raising (2.17) to the g th power and multiplying by $\rho(z)$ from the right we will have

$$\begin{aligned} \left(\iint_{G^2} \rho(yxy^{-1}x^{-1}) dx dy \right)^g \rho(z) &= \frac{1}{\chi(1)^{2g}} \rho(z) \\ \underbrace{\int_G \cdots \int_G}_{2g \text{ times}} \rho([y_1, x_1] \cdots [y_g, x_g] z) dx_1 dy_1 \cdots dx_g dy_g &= \frac{1}{\chi(1)^{2g}} \rho(z). \end{aligned} \quad (2.18)$$

Taking trace at both sides yet one more time we have:

$$\int_G \cdots \int_G \chi([y_1, x_1] \cdots [y_g, x_g] z) dx_1 dy_1 \cdots dx_g dy_g = \frac{\chi(z)}{\chi(1)^{2g}}. \quad (2.19)$$

Multiplying this by $\chi(1)$ and summing over all $\chi \in \text{Irr}(G)$ we see that the sum

$$\sum_{\chi \in \text{Irr}(G)} \frac{\chi(z)}{\chi(1)^{2g-1}} \quad (2.20)$$

is equal to

$$\int_G \cdots \int_G \sum_{\chi \in \text{Irr}(G)} \chi(1) \chi([y_1, x_1] \cdots [y_g, x_g] z) dx_1 dy_1 \cdots dx_g dy_g \quad (2.21)$$

and thanks to the orthogonality relations from (2.11) only those terms with $[y_1, x_1] \cdots [y_g, x_g] z = 1$ survive in (2.21), and we get

$$\frac{1}{|G|^{2g-1}} \# \{ [y_1, x_1] \cdots [y_g, x_g] z = 1 \} = \sum_{\chi \in \text{Irr}(G)} \frac{\chi(z)}{\chi(1)^{2g-1}} \quad (2.22)$$

from which the proposition follows immediately. \square

Remark 2.2.10. In Chapter 5 we will compute, with the aid of formula (2.12), the number of \mathbb{F}_q -points in the twisted Sl_n -character varieties, mimicking the ideas of [9] for the Gl_n -case.

As a particular case of Proposition 2.2.1, for $z = 1$ (the identity element of G) we recover the well known formula (see [5] for instance).

Corollary 2.2.2. *Let Σ_g be a genus g compact Riemann surface, π_1 its fundamental group and G a finite group. Then*

$$\frac{1}{|G|} \# \text{Hom}(\pi_1(\Sigma_g), G) = \sum_{\chi \in \text{Irr}(G)} \left(\frac{|G|}{\chi(1)} \right)^{2g-2}. \quad (2.23)$$

2.2.3 Clifford Theory

To apply formula 2.2.1, we will need the character table of $\text{Sl}_n(q)$, computed by Lehrer in [12]. In order to deal with these characters we will also make use of the results of [11].

The standard setting for Clifford Theory is the following:

G is a finite group and H is a normal subgroup of G such that the quotient G/H is abelian.

There is a natural action of G on $\text{Irr}(H)$ given by the conjugation

$$(g\theta)(h) := \theta(g^{-1}hg).$$

Since the characters of H are class functions, H acts trivially on $\text{Irr}(H)$, therefore this G -action induces one of G/H .

We note $[\theta]$ the class of $\theta \in \text{Irr}(H)$ under this action.

There is also a natural action of $\widehat{(G/H)} := \text{Hom}(G/H, \mathbb{C}^\times)$, the character group of G/H , on $\text{Irr}(G)$ given by multiplication

$$(\psi\chi)(g) := \psi([g])\chi(g)$$

where $\psi \in \widehat{(G/H)}$ and $[g] \in G/H$ is the class of $g \in G$.

We note $[\chi]$ the orbit of $\chi \in \text{Irr}(G)$ under this action.

Theorem 2.2.3. (1 from [11]) *Let $\chi, \chi' \in \text{Irr}(G)$. Then:*

- $\chi_H = \chi'_H$ if and only if $\chi' \in [\chi]$.
- $|\text{Stab}(\chi)| = \langle \chi_H, \chi_H \rangle$.
- χ_H is irreducible if and only if $|\text{Stab}(\chi)| = 1$.
- if $\theta \in \text{Irr}(H)$ is a constituent of χ_H , then there is a positive integer $e_H(\chi)$ such that

$$\chi_H = e_H(\chi) \sum_{\theta' \in [\theta]} \theta'$$

and

$$e_H(\chi)^2 |[\chi]| |[\theta]| = |G/H|.$$

- $e_H(\chi) = 1$ for G/H cyclic.

Remark 2.2.11. When the quotient G/H is cyclic, theorem 2.2.3 establishes a canonical bijection between both sets of orbits

$$\text{Irr}(G)/\widehat{(G/H)} \quad \text{and} \quad \text{Irr}(H)/(G/H)$$

by which an orbit $[\chi]$ is mapped to the G/H -orbit $[\theta]$ defined as the set of irreducible components of $\chi_H = \sum_{\theta' \in [\theta]} \theta'$, and the product of the sizes of both orbits is $|G/H|$.

2.2.4 Characters of finite General Linear Groups

Thanks to Clifford Theory and motivated by the formulas for the PGL_n -character varieties from [9], we need a description of the irreducible characters of the finite general linear groups $\mathrm{GL}_n(q)$.

They were computed by Green in [7]. We will follow Macdonald's approach from [13], with some notation taken from [18].

Let us fix a prime power $q = p^m$. We note \mathbb{F}_{q^r} the field of q^r elements inside a fixed algebraic closure $\overline{\mathbb{F}}_q$ of \mathbb{F}_q .

Notation. Let $Frob_q : x \mapsto x^q$ be the Frobenius automorphism.

Definition 2.2.3. Clearly \mathbb{F}_{q^r} is $\overline{\mathbb{F}}_q^{Frob_q^r}$ the field of elements fixed by $Frob_q^r$.

Notation. Whenever $n|m$ we have the norm map $N_n^m : \mathbb{F}_{q^m} \twoheadrightarrow \mathbb{F}_{q^n}$ which is known to be surjective.

Definition 2.2.4. Let $\Gamma_n := \mathrm{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{C}^\times)$ the character group of the cyclic group $\mathbb{F}_{q^n}^\times$. By Remark 2.2.5, the transpose of the norm gives us an injective map

$${}^T N_n^m : \Gamma_n \hookrightarrow \Gamma_m \tag{2.24}$$

when $n|m$, defining a direct system whose colimit we note

$$\Gamma := \varinjlim \Gamma_n. \tag{2.25}$$

Remark 2.2.12. The Frobenius automorphism induces by transposition, a map on Γ given by $\gamma \mapsto \gamma^q$ and just as before we can identify Γ_n with $\Gamma^{Frob_q^n}$, the elements of Γ fixed by $Frob_q^n$.

Definition 2.2.5. The degree of $\gamma \in \Gamma$ is $\deg(\gamma) \in \mathbb{N}$ the size of the Frobenius orbit of γ .

Remark 2.2.13. Since we are identifying Γ_n with elements of the colimit Γ , given some $\gamma \in \Gamma$ and an element $\xi \in \mathbb{F}_q^\times$ there are in general more than one way to interpret the evaluation $\gamma(\xi)$, for if γ belongs to $\Gamma_n = \text{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{C}^\times)$ there is another representative ${}^T N_n^{nk}(\gamma)$ for γ in every Γ_{nk} , and the evaluation of the new representative at ξ gives ${}^T N_n^{nk}(\gamma)(\xi) = \gamma(N_n^{nk}(\xi)) = \gamma(\xi^k) = \gamma(\xi)^k$.

To avoid confusion we will say in which character group Γ_n the character is supposed to be considered.

Remark 2.2.14. For instance, given $\gamma \in \Gamma_d = \text{Hom}(\mathbb{F}_{q^d}^\times, \mathbb{C}^\times)$ the product

$${}^T N_1^d(\gamma) := \gamma \gamma^q \gamma^{q^2} \dots \gamma^{q^{d-1}}$$

is Frob_q -stable, hence there is a representative of it in $\Gamma_1 = \text{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$. Let us call such representative $\gamma_1 \in \text{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$.

Evaluation at any $\xi \in \mathbb{F}_q^\times$ of ${}^T N_1^d(\gamma)$ in Γ_d will be

$$\gamma(\xi) \gamma(\xi^q) \dots \gamma(\xi^{q^{d-1}}) = \gamma(\xi)^d$$

since $\xi^q = \xi$.

On the other hand, evaluation in Γ_1 (i.e.: computing $\gamma_1(\xi)$) amounts to evaluating at ξ the restriction to \mathbb{F}_q^\times of any of the Frob_q -conjugates of γ , since all such conjugates agree on \mathbb{F}_q^\times and the restriction $\gamma|_{\mathbb{F}_q^\times} \in \Gamma_1$ will become ${}^T N_1^d(\gamma)$ under the inclusion $\Gamma_1 \hookrightarrow \Gamma_d$.

Definition 2.2.6. Consider $\Theta := \Gamma /_{Frob_q}$ the collection of $Frob_q$ -orbits of Γ .

Remark 2.2.15. Every $Frob_q$ -orbit $\{\gamma\}$ for $\gamma \in \Gamma$ is finite.

Remark 2.2.16. By last remark, the set \mathcal{P}^Θ of Θ -partitions Λ is defined, and so are

- the size $|\Lambda| = \sum_{\gamma \in \Gamma} |\Lambda(\gamma)|$,
- the conjugate $\Lambda'(\gamma) = \Lambda(\gamma)' \in \mathcal{P}$,
- the multiplicities

$$m_{d,\lambda} = m_{d,\lambda}(\Lambda) = \# \{ \{\gamma\} \mid \deg(\gamma) = d, \Lambda(\gamma) = \lambda \},$$

- the type $\tau(\Lambda) = (m_{d,\lambda})_{d \geq 1, \lambda \in \mathcal{P}}$,
- the hook polynomials H_Λ , H_τ and
- the set \mathcal{P}_m^Θ of multi-partitions size m , for $m \geq 0$.

Theorem 2.2.4. ((6.8) from [13] page 286) *There is a bijective correspondence between \mathcal{P}_n^Θ and the irreducible characters of $Gl_n(q)$ by which the character χ_Λ corresponding to Λ has degree*

$$\chi_\Lambda(1) = \frac{\prod_{i=1}^n (q^i - 1)}{q^{-n(\Lambda')} H_\Lambda(q)} \quad (2.26)$$

and the value at the central element ζId ($\zeta \in \mathbb{F}_q^$) is given by*

$$\chi_\Lambda(\zeta Id) = \Delta_\Lambda(\zeta) \chi_\Lambda(1) \quad (2.27)$$

where

$$\Delta_\Lambda = \prod_{\gamma \in \Gamma} \gamma^{|\Lambda(\gamma)|} \in \Gamma_1. \quad (2.28)$$

Notation. We write $\Delta_\chi := \Delta_\Lambda$ for $\chi = \chi_\Lambda$, the character $\chi \in \text{Irr}(G)$ associated to the function $\Lambda \in \mathcal{P}_n^\Theta$.

Remark 2.2.17. Since Δ_Λ is $Frob_q$ -stable, the evaluation of (2.27) should be done as explained in Remark 2.2.14, meaning

$$\begin{aligned} \Delta_\Lambda(\zeta) &= \prod_{\gamma \in \Gamma} \gamma^{|\Lambda(\gamma)|}(\zeta) \\ &= \prod_{\{\gamma\}} \gamma^{|\Lambda(\gamma)|}(\zeta) \end{aligned} \quad (2.29)$$

where last product is taken over the $Frob_q$ -orbits $\{\gamma\}$ and each evaluation $\gamma(\zeta)$ is done in $\Gamma_{\deg(\gamma)}$.

Remark 2.2.18. The trivial character 1_G corresponds to the Θ -partition $\Lambda \in \mathcal{P}_n^\Theta$ sending $1 \in \Gamma_1$ to $(1, \dots, 1)$ of size n (and hence mapping all other characters to \emptyset).

Remark 2.2.19. More generally, from (2.26) follows that all 1-dimensional representations of $\text{Gl}_n(q)$ are in one-to-one correspondence with the multi-partitions $\Lambda \in \mathcal{P}_n^\Theta$ of type τ_p with $(\tau_p)_{1,(1,\dots,1)} = 1$.

Remark 2.2.20. Since the degree of a character χ_Λ depends only on the type $\tau = \tau(\Lambda)$ we write $\chi_\tau(1) := \chi_\Lambda(1)$ and from 2.26 we get

$$\frac{|Gl_n(q)|}{\chi_\tau(1)} = (-1)^n q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q) = q^{\binom{n}{2} - n(\Lambda')} H_\Lambda(q) \quad (2.30)$$

which actually lies in $\mathbb{Z}[q]$.

Definition 2.2.7. The \circ -product of characters χ_1 from $\mathrm{Gl}_{n_1}(q)$ and χ_2 from $\mathrm{Gl}_{n_2}(q)$ is defined by considering the parabolic subgroup P of upper triangular block matrices of the form:

$$\begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$$

with $A_i \in \mathrm{Gl}_{n_i}(q)$ and its character $\chi_1 * \chi_2$ constructed from $\chi_1 \otimes \chi_2$ by composition with the natural projection

$$P \twoheadrightarrow \mathrm{Gl}_{n_1}(q) \times \mathrm{Gl}_{n_2}(q).$$

Then take

$$\chi_1 \circ \chi_2 := (\chi_1 * \chi_2)^G = \mathrm{Ind}_P^G(\chi_1 * \chi_2)$$

the induced character.

Remark 2.2.21. In [7], the character $\chi_\Lambda \in \mathrm{Irr}(\mathrm{Gl}_n(q))$ is constructed for $\Lambda \in \mathcal{P}_n^\Theta$ as

$$\chi_\Lambda = J^{\langle \gamma_1 \rangle}(\Lambda(\gamma_1)) \circ \dots \circ J^{\langle \gamma_m \rangle}(\Lambda(\gamma_m))$$

where $J^{\langle \gamma \rangle}(\lambda)$ is certain simple character of $\mathrm{Gl}_{\deg(\gamma)|\lambda|}(q)$.

Remark 2.2.22. Since the \circ -product is easily seen to be associative and commutative, the character $J^{\langle \gamma_1 \rangle}(\Lambda(\gamma_1)) \circ \dots \circ J^{\langle \gamma_m \rangle}(\Lambda(\gamma_m))$ is well defined.

2.2.5 The actions σ_α and τ_α

The determinant identifies the quotient $\mathrm{Gl}_n(q)/\mathrm{Sl}_n(q)$ with the cyclic group \mathbb{F}_q^\times .

The $\widehat{\mathbb{F}}_q^\times$ -action on $\text{Irr}(\text{Gl}_n(q))$ we talked about in Section 2.2.3 corresponds to the σ_α operation defined for $\alpha \in \widehat{\mathbb{F}}_q^\times$ in [11] as

$$\sigma_\alpha(\chi)(x) := \alpha(\det(x))\chi(x).$$

There is another action defined in [12]. Namely the τ_α action defined by

$$\tau_\alpha(\chi) := J^{\langle \alpha \gamma_1 \rangle}(\Lambda(\gamma_1)) \circ \dots \circ J^{\langle \alpha \gamma_m \rangle}(\Lambda(\gamma_m))$$

where the product $\alpha \gamma_i$ should be computed in Γ_{d_i} (following Remark 2.2.14).

Remark 2.2.23. The operation $\chi \mapsto \tau_\alpha(\chi)$ from [12] in the language of Θ -partitions is given by

$$\Lambda \longmapsto \Lambda \alpha^{-1}$$

where

$$\Lambda \alpha^{-1}(\gamma) := \Lambda(\alpha^{-1} \gamma).$$

Remark 2.2.24. The proposition of page 133 in [11] asserts that both operations τ_α and σ_α coincide. This will help us to regard $\text{Irr}(\text{Gl}_n(q))$ as a set of functions, and $\text{Irr}(\text{Sl}_n(q))$ as a set of orbits of those functions under the recently described action.

Theorem 2.2.3 and last Remark together imply the following

Theorem 2.2.5 (4 in [12], 2 in [11]). *Let $\chi, \chi' \in \text{Irr}(\text{Gl}_n(q))$, then their restrictions $\chi_{\text{Sl}_n(q)}$ and $\chi'_{\text{Sl}_n(q)}$ agree if and only if there exists $\alpha \in \widehat{\mathbb{F}}_q^\times$ such that $\tau_\alpha(\chi) = \chi'$.*

Chapter 3

The E -polynomial of a Character Variety

3.1 Character Varieties

In this Section we define the family of Character Varieties we want to study, and list some of their properties.

Definition 3.1.1. Let n and g be positive integers, \mathbb{K} a field possessing a primitive n th root of unity we call ζ_n , and $Id \in \mathrm{Sl}_n(\mathbb{K})$ the identity matrix. The twisted Sl_n -character variety $\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{K}))$ is given by all the classes of $2g$ -tuples $(A_1, B_1, \dots, A_g, B_g)$ in $\mathrm{Sl}_n(\mathbb{K})^{2g}$ satisfying the following matricial equation

$$[A_1, B_1] \dots [A_g, B_g] = \zeta_n Id \quad (3.1)$$

modulo $\mathrm{PGL}_n(\mathbb{K})$ -conjugation. In other words, it is the geometric quotient

$$\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{K})) := \{[A_1, B_1] \dots [A_g, B_g] = \zeta_n Id\} // \mathrm{PGL}_n(\mathbb{K}) \quad (3.2)$$

where $A_i, B_i \in \mathrm{Sl}_n(\mathbb{K})$.

Remark 3.1.1. They appeared in Theorem 2.2.12 of [9], to prove that the PGL_n -character variety $\mathcal{M}^g(\mathrm{PGL}_n)$ is an orbifold.

Remark 3.1.2. For a finite field \mathbb{F}_q to have a primitive n th root of unity it is necessary and sufficient that n divides $q - 1$.

Notation. When $\mathbb{K} = \mathbb{F}_q$ is the finite field with q elements we shall write $\mathcal{M}^g(\mathrm{Sl}_n)(q)$ for $\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q))$.

Remark 3.1.3. The twisted Sl_n -character varieties are non-singular and equidimensional (see page 570 of [9]).

Proposition 3.1.1. *Let $(A_1, B_1, \dots, A_g, B_g) \in \mathrm{Sl}_n(\mathbb{K})^{2g}$ satisfying (3.1), and $Z \in \mathbb{K}^{n \times n}$ a matrix commuting with all the A_i 's and B_i 's. Then Z is a scalar matrix (i.e.: a scalar multiple of the identity Id).*

Proof. The only nonzero subspace $K \subseteq \mathbb{K}^n$ invariant by the A_i 's and B_i 's is \mathbb{K}^n , since restriction to such K will lead to the identity

$$[A_1|_K, B_1|_K] \dots [A_g|_K, B_g|_K] = \zeta_n Id_K \quad (3.3)$$

which, by taking determinants, gives $1 = \zeta_n^{\dim_{\mathbb{K}} K}$, implying $\dim_{\mathbb{K}} K = n$ and then $K = \mathbb{K}^n$.

Extending \mathbb{K} if necessary, we can assume Z has an eigenvalue $\lambda \in \mathbb{K}$. Since the eigenspace $K = \mathrm{Ker}(Z - \lambda Id)$ is going to be invariant by multiplication by the A_i 's and B_i 's, we conclude that $K = \mathbb{K}^n$, or what is the same, $Z = \lambda Id$. \square

Remark 3.1.4. The stabilizer of a $2g$ -tuple $(A_1, B_1, \dots, A_g, B_g) \in \mathrm{Sl}_n(\mathbb{K})^{2g}$ satisfying (3.1) under the $\mathrm{Gl}_n(\mathbb{K})$ -action of simultaneous conjugation is, according to proposition (3.1.1), the center of $\mathrm{Gl}_n(\mathbb{K})$.

Therefore we have the following two corollaries

Corollary 3.1.2. *The induced $\mathrm{PGL}_n(\mathbb{K})$ -action results free.*

Corollary 3.1.3. *When $\mathbb{K} = \mathbb{F}_q$, the number of \mathbb{K} -points of $\mathcal{M}^g(\mathrm{Sl}_n)(q)$ will be the number of solutions of (3.1) in $\mathrm{Sl}_n(q)$ divided by $|\mathrm{PGL}_n(q)|$.*

3.2 Mixed Hodge Structures

In [2] and [3] Deligne proved the existence of the following structure. Let X be a complex algebraic variety. For each j the cohomology group $H^j(X, \mathbb{Q})$ and its complexification $H^j(X, \mathbb{C})$ have filtrations

$$0 = W_{-1} \subseteq W_0 \subseteq \dots \subseteq W_{2j} = H^j(X, \mathbb{Q}) \quad (3.4)$$

and

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^m \supseteq F^{m+1} = 0 \quad (3.5)$$

called weight and Hodge filtration, respectively, with the property that the filtration induced by F on $\mathrm{gr}_l W := W_l / W_{l-1}$ (the complexification of the graded pieces of the weight filtration) equips every graded piece with a pure Hodge structure of weight l .

Remark 3.2.1. Such a structure is called the Mixed Hodge Structure, or MHS, of X . It behaves nicely with respect to the most common operations on cohomology. For instance, it is preserved by maps

$$f^* : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$$

induced from algebraic maps $f : X \rightarrow Y$, by maps induced from field auto-

morphisms $\sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q})$, by Künneth isomorphism

$$H^*(X \times Y, \mathbb{Q}) \cong H^*(X, \mathbb{Q}) \otimes H^*(Y, \mathbb{Q})$$

and by cup products.

For smooth X we get trivial $W_{j-1}H^j(X)$.

Remark 3.2.2. There is also a Mixed Hodge Structure for cohomology with compact support $H_c^*(X, \mathbb{Q})$, and the interplay with $H^*(X, \mathbb{Q})$ given by Poincaré duality respects both MHS.

Definition 3.2.1. Define:

$$h^{p,q;j}(X) := \dim_{\mathbb{C}} \text{gr}_p F \text{gr}_{p+q} W H^j(X, \mathbb{C}), \quad (3.6)$$

and

$$h_c^{p,q;j}(X) := \dim_{\mathbb{C}} \text{gr}_p F \text{gr}_{p+q} W H_c^j(X, \mathbb{C}) \quad (3.7)$$

the Mixed Hodge numbers and the compactly supported Mixed Hodge numbers, respectively.

These numbers can be encoded in the following polynomials:

Definition 3.2.2. The Mixed Hodge Polynomial

$$H(x, y, t; X) := \sum h^{p,q;j}(X) x^p y^q t^j, \quad (3.8)$$

the compactly supported Mixed Hodge Polynomial

$$H_c(x, y, t; X) := \sum h_c^{p,q;j}(X) x^p y^q t^j, \quad (3.9)$$

and finally let

$$E(x, y; X) := H_c(x, y, -1; X) \quad (3.10)$$

be the E -polynomial of X .

Remark 3.2.3. From the definition of $E(x, y; X)$, (3.7), (3.4) and (3.5) we have:

- $E(1, 1; X)$ is the Euler Characteristic of X
- The total degree of $E(x, y; X)$ is twice the dimension of X .
- The coefficient of $x^{\dim(X)} y^{\dim(X)}$ in $E(x, y; X)$ is the number of connected components of X .

3.3 Katz's Theorem

Definition 3.3.1. For a complex algebraic variety X , a finitely generated \mathbb{Z} -algebra R and a fixed embedding

$$\phi : R \hookrightarrow \mathbb{C}$$

we say that a separated scheme \mathfrak{X}/R is a spreading out of X if its extension of scalars \mathfrak{X}_ϕ is isomorphic to X .

Remark 3.3.1. A spreading out is an object that links a complex algebraic variety X with its finite field counterparts.

Definition 3.3.2. Let us now assume X has a spreading out \mathfrak{X} such that for every ring homomorphism $\psi : R \rightarrow \mathbb{F}_q$, the number of points of $\mathfrak{X}_\psi(\mathbb{F}_q)$ is given

by $P_X(q)$ for some fixed $P_X(t) \in \mathbb{Z}[t]$, independent of ψ . We say that this X has polynomial count and that P_X is the counting polynomial.

The following result holds:

Theorem 3.3.1. *[(2.1.8) in [9] proved by Katz in Appendix] Let X be a variety over \mathbb{C} . Assume X has polynomial count with polynomial $P_X(t) \in \mathbb{Z}[t]$, then the E -polynomial of X is given by $E(x, y; X) = P_X(xy)$.*

Notation. In such case, we write

$$E(q; X) := E(\sqrt{q}, \sqrt{q}; X).$$

Remark 3.3.2. This polynomial $E(q; X) \in \mathbb{Z}[q]$ agrees with the counting polynomial $P_X(q)$ and encodes all the information that $E(x, y; X)$ does.

Remark 3.3.3. For X a variety with polynomial count given by $P_X(q)$, by Remark 3.2.3 we have:

- The Euler characteristic of X can be computed as $P_X(1)$.
- The dimension $\dim(X)$ is the degree of $P_X(q)$.
- The principal coefficient of $P_X(q)$ is the number of connected components of X .

3.4 Spreading out of $\mathcal{M}^g(\mathrm{Sl}_n)$

Let us fix $\zeta_{2n} \in \mathbb{C}$ and $\zeta_n = \zeta_{2n}^2$ primitive $2n$ th and n th-roots of 1, respectively.

Let R be the finitely generated \mathbb{Z} -algebra $Z[\zeta_{2n}, \frac{1}{n}] \subseteq \mathbb{C}$.

Consider $\widehat{\mathfrak{M}}^g(\mathrm{Sl}_n)$ the closed affine subscheme of \mathfrak{S}_n^{2g} over R defined by the matricial equation (3.1).

Definition 3.4.1. Define $\mathfrak{M}^g(\mathrm{Sl}_n)$ as the categorical quotient:

$$\mathfrak{M}^g(\mathrm{Sl}_n) := \mathrm{Spec}(R[\widehat{\mathfrak{M}}^g(\mathrm{Sl}_n)]^{\mathrm{PGL}_n(R)}). \quad (3.11)$$

Remark 3.4.1. By flatness of $R \hookrightarrow \mathbb{C}$ and Lemma 2 from [16] we have

$$R[\widehat{\mathfrak{M}}^g(\mathrm{Sl}_n)]^{\mathrm{PGL}_n(R)} \otimes_R \mathbb{C} = \mathbb{C}[\widehat{\mathfrak{M}}^g(\mathrm{Sl}_n)]^{\mathrm{PGL}_n(\mathbb{C})}. \quad (3.12)$$

Then $\mathfrak{M}^g(\mathrm{Sl}_n)$ is a spreading out of $\mathfrak{M}^g(\mathrm{Sl}_n)$.

Remark 3.4.2. By Lemma 3.2 from the appendix of [10], the $\widehat{\mathfrak{M}}^g(\mathrm{Sl}_n)$ -fibers of an \mathbb{F}_q -point of $\mathfrak{M}^g(\mathrm{Sl}_n)$ will be nonempty $\mathrm{PGL}_n(\mathbb{F}_q)$ -orbits, hence

$$\mathfrak{M}^g(\mathrm{Sl}_n)(\mathbb{F}_q) = \widehat{\mathfrak{M}}^g(\mathrm{Sl}_n(\mathbb{F}_q)) / \mathrm{PGL}_n(\mathbb{F}_q) \quad (3.13)$$

and by Corollary 3.1.2 the action will be free.

Remark 3.4.3. Last Remark, together with Corollary 3.1.3 imply that the counting functions for our character varieties $\mathfrak{M}^g(\mathrm{Sl}_n)$ are

$$N_n^g(q) := \frac{1}{|\mathrm{PGL}_n(\mathbb{F}_q)|} \tilde{N}_n^g(q) \quad (3.14)$$

where

$$\tilde{N}_n^g(q) = \sum_{\theta \in \mathrm{Irr}(\mathrm{Sl}_n(\mathbb{F}_q))} \left(\frac{|\mathrm{Sl}_n(\mathbb{F}_q)|}{\theta(1)} \right)^{2g-1} \theta(\zeta_n Id). \quad (3.15)$$

is the number of solutions of (3.1) (thanks to Proposition 2.2.1).

Remark 3.4.4. Note that since $R \subseteq \mathbb{C}$ is defined as the finite \mathbb{Z} -algebra $Z[\zeta_{2n}, \frac{1}{n}]$, every ring homomorphism to a field F

$$\phi : R \rightarrow F$$

must necessarily send n to a unit.

Since

$$(1 - \zeta_{2n})(1 - \zeta_{2n}^2) \dots (1 - \zeta_{2n}^{2n-1}) = 2n$$

when n is even, $2n$ must also be mapped to a unit, and hence ζ_{2n} is mapped to an element of order $2n$. In particular, a primitive n th root of unity ζ_n is mapped to an element of order n .

When n is odd, 2 could be in the kernel of ϕ , making F a characteristic 2 field (in which case no element has even order). A similar reasoning with

$$(1 - \zeta_n)(1 - \zeta_n^2) \dots (1 - \zeta_n^{n-1}) = n$$

proves that ζ_n is mapped to an order n element.

Hence, in both cases we have an element of order n in the target field.

When $F = \mathbb{F}_q$ is a finite field with q elements, F has an element of order n if and only if n divides $q - 1$. When n is even, F must also have an order $2n$ element, making $q - 1$ divisible by $2n$.

Summarizing, under this setting, our counting functions only make sense for those q satisfying 1.9.

Remark 3.4.5. If we took $R = \mathbb{Z}[\zeta_n, \frac{1}{n}]$ instead, most of the above follows, except that the only condition for q is to be congruent to 1 modulo n . This is all what we need for $\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q))$ to make sense. The counting function for this spreading out will not in general be a polynomial (as we will see in Chapter 4), and that is why we take R with a primitive $2n$ th root of 1.

Chapter 4

The Case $n = 2$

In this Chapter we are going to list the characters of $\mathrm{Gl}_2(q)$ and see how their restrictions to $\mathrm{Sl}_2(q)$ split for odd q , giving all the irreducible characters of such subgroup as we saw in 2.2.3. We then use these characters to compute the number of points of $\mathcal{M}^g(\mathrm{Sl}_2)(q)$ for q odd, and hence their E -polynomials (by Theorem 3.3.1). We finally get some geometric and topological information thanks to Remark 3.3.3.

4.1 Character Table of $\mathrm{Gl}_2(q)$

As shown in [4] and [6], the character table of $\mathrm{Gl}_2(q)$ is:

Table 4.1: Character Table of $\mathrm{Gl}_2(q)$.

| classes | $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ | $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ | $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ | $\begin{pmatrix} x & 0 \\ 0 & x^q \end{pmatrix}$ |
|------------------------|--|--|--|--|
| # of classes | $q - 1$ | $q - 1$ | $(q - 1)(q - 2)/2$ | $q(q - 1)/2$ |
| class size | 1 | $q^2 - 1$ | $q(q + 1)$ | $q(q - 1)$ |
| $R_T^G(\alpha, \beta)$ | $(q + 1)\alpha(a)\beta(a)$ | $\alpha(a)\beta(a)$ | $\alpha(a)\beta(b) + \alpha(b)\beta(a)$ | 0 |
| $-R_{T^s}^G(\omega)$ | $(q - 1)\omega(a)$ | $-\omega(a)$ | 0 | $-(\omega(x) + \omega(x^q))$ |
| $\sigma_\alpha(1_G)$ | $\alpha(a^2)$ | $\alpha(a^2)$ | $\alpha(ab)$ | $\alpha(x^{q+1})$ |
| $\sigma_\alpha(St_G)$ | $q\alpha(a^2)$ | 0 | $\alpha(ab)$ | $-\alpha(x^{q+1})$ |

where $a, b \in \mathbb{F}_q^\times$, $a \neq b$, $x \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, $\alpha, \beta \in \widehat{\mathbb{F}}_q^\times$, $\alpha \neq \beta$, $\omega \in \widehat{\mathbb{F}}_{q^2}^\times$, $\omega \neq \omega^q$, St_G

is the Steinberg representation and 1_G the trivial one.

The class $\begin{pmatrix} x & 0 \\ 0 & x^q \end{pmatrix}$ corresponds to those matrices in $\text{Gl}_2(q)$ having eigenvalues x and x^q , so that they are diagonalizable in \mathbb{F}_{q^2} but not in \mathbb{F}_q .

The rows correspond to the different types τ_1, τ_2, τ_3 and τ_4 of size 2, all of them having a unique nonzero multiplicity $(\tau_1)_{1,(1)} = 2, (\tau_2)_{2,(1)} = 1, (\tau_3)_{1,(2)} = 1$ and $(\tau_4)_{1,(1,1)} = 1$ respectively.

- Type τ_1 has two characters $\alpha \neq \beta$ of degree 1 mapped to the partition $(1) \in \mathcal{P}_1$ and the notation $R_T^G(\alpha, \beta)$ corresponds to Deligne-Lusztig [1]. We must point out that $R_T^G(\alpha, \beta) \cong R_T^G(\beta, \alpha)$.
- Analogously, type τ_2 (one degree 2 character $\omega \in \Gamma_2$ mapped to $(1) \in \mathcal{P}_1$) corresponds to $-R_{T^s}^G(\omega)$ where T^s is the non-split torus (see [1]). In this case we have $-R_{T^s}^G(\omega) \cong -R_{T^s}^G(\omega^q)$.
- For every $\alpha \in \Gamma_1 = \widehat{\mathbb{F}_q}^\times$, the composition with the determinant $\det : \text{Gl}_n(q) \rightarrow \mathbb{F}_q^\times$ gives a new character we noted $\sigma_\alpha(1_G)$ (defined in Subsection 2.2.5). Type τ_3 is the collection of all these.
- Type τ_4 consists on all the $\sigma_\alpha(St_G)$ found by tensoring one character of τ_3 with the Steinberg representation.

4.2 Character Table of $\text{Sl}_2(q)$

To compute the character table of $\text{Sl}_2(q)$ we will apply the results from Subsections 2.2.3 and 2.2.5. Let us now assume q odd.

The action of $\widehat{\mathbb{F}}_q^\times$ on $\text{Irr}(\text{Gl}_2(q))$ identifies its orbits with representations of $\text{Sl}_2(q)$ (by restriction) each of which is a sum of representations in an orbit of the \mathbb{F}_q^\times -action given by conjugation (under the identification $\mathbb{F}_q^\times \cong \text{Gl}_2(q)/\text{Sl}_2(q)$). And the product of the sizes of corresponding orbits must be $q - 1$.

Remark 4.2.1. The characters of type τ_3 have all dimension 1 and hence, their restrictions to $\text{Sl}_2(q)$ have to be irreducible. In fact, all of them restrict to the trivial representation of $\text{Sl}_2(q)$.

Remark 4.2.2. The type τ_4 is a full $\widehat{\mathbb{F}}_q^\times$ -orbit of size $q - 1$ (under the action from Section 2.2.3), so the restriction to $\text{Sl}_2(q)$ of every character of this type will remain irreducible according to Remark 2.2.11.

Remark 4.2.3. The action on a representation $R_T^G(\alpha, \beta)$ of type τ_1 identifies the unordered pairs $\{\alpha, \beta\}$ with all pairs $\{\gamma\alpha, \gamma\beta\}$ for $\gamma \in \widehat{\mathbb{F}}_q^\times$, in particular with $\{\alpha/\beta, 1\}$. To simplify, we may assume the character is $R_T^G(\alpha, 1)$, for a nontrivial α . The orbit of $\{\alpha, 1\}$ is made up of all the unordered pairs $\{\gamma\alpha, \gamma\}$.

Its stabilizer is trivial unless $\{\gamma\alpha, \gamma\} = \{\alpha, 1\}$ for some $\gamma \neq 1$. That can only happen for $\gamma = \alpha^{-1}$ and $\alpha^2 = 1$. Let us call $\alpha_0 \in \widehat{\mathbb{F}}_q^\times$ the unique order two character. There is only one since the group $\widehat{\mathbb{F}}_q^\times \cong \mathbb{F}_q^\times$ is cyclic.

Remark 4.2.4. We must point out that since both $\{\alpha, 1\}$ and $\{1, \alpha^{-1}\}$ lie in the same orbit, $R_T^G(\alpha, 1)$ and $R_T^G(\alpha^{-1}, 1)$ restrict to the same representation of $\text{Sl}_2(q)$, so we only have $\frac{q-3}{2}$ of this type ($\alpha^2 \neq 1$).

In conclusion:

Proposition 4.2.1. *Every character of type τ_2 restricts to an irreducible character of $\text{Sl}_2(q)$, except that of $R_T^G(\alpha_0, 1)$ whose restriction has two irreducible components. We will call them $\chi_{\alpha_0}^+$ and $\chi_{\alpha_0}^-$.*

Remark 4.2.5. With $-R_{T^s}^G(\omega)$ the situation is similar. The orbit of a representation corresponding to ω is given by those representations associated to the $\gamma\omega \in \Gamma_2$ where $\gamma \in \Gamma_1$ is regarded as in Γ_2 via the inclusion from (2.24)

$${}^T N_1^2 : \Gamma_1 \hookrightarrow \Gamma_2.$$

Therefore, $(\omega\gamma)(a)$ should be computed as $\omega(a)\gamma(N_1^2(a))$ (which agrees with $\omega(a)\gamma(a^{q+1}) = \omega(a)\gamma(a^2) = \omega(a)\gamma^2(a)$ for $a \in \mathbb{F}_q$ in accordance with Remark 2.2.14).

Since $\widehat{\mathbb{F}}_q^\times$ acts by multiplication, we have $q - 1$ different characters $\gamma\omega$. But the irreducible representation $-R_{T^s}^G(\omega)$ depends only on the Frobenius orbit $\{\omega, \omega^q\}$, hence for a fixed $\omega \in \Gamma_2$ the stabilizer of the $\widehat{\mathbb{F}}_q^\times$ -action on the representations $-R_{T^s}^G(\gamma\omega)$ is given by the trivial character and all the $\gamma \in \Gamma_1$ such that $\omega\gamma = \omega^q$. And this can only happen when $\omega\gamma^2 = \omega$ (since $\omega^{q^2} = \omega$ and $\gamma^q = \gamma$). Therefore, for $-R_{T^s}^G(\omega)$ to be stable by a nontrivial γ we must have $\gamma = \alpha_0$ (the character of order 2 from Remark 4.2.3) and $\omega^2 \in \Gamma_1$ by Remark 2.2.12 and

$$(\omega^2)^q = (\omega^q)^2 = (\omega\alpha_0)^2 = \omega^2.$$

Therefore, we conclude the following

Proposition 4.2.2. *The representation $-R_{T^s}^G(\omega)$ will restrict to an irreducible representation of $\mathrm{Sl}_2(q)$ unless ω is the square root of a (non-square) degree one character, in which case its restriction have two components.*

Remark 4.2.6. The map $\omega \mapsto \omega^{q-1}$ identifies characters $\omega, \gamma\omega$ in the same orbit and its kernel is Γ_1 . Its image is the subgroup of Γ_2 of order $q+1$. By Remark 2.2.5 we can regard it as the dual of a cyclic group of order $q+1$ which we note μ_{q+1} .

Remark 4.2.7. The characters ω_0 whose square have degree one will be mapped to those characters of order 1 or 2.

By an abuse of notation we will keep calling them $\omega, \omega_0 \in \mu_{q+1}$, $-R_{T^s}^G(\omega)$ is the restriction to $\mathrm{Sl}_2(q)$ of any representation $-R_{T^s}^G(\gamma\omega)$ of $\mathrm{Gl}_2(q)$. These are still irreducible for $\omega^2 \neq 0$, and for the nontrivial $\omega_0 \in \mu_{q+1}$ with $\omega_0^2 = 1$ we have two components and note its characters $\chi_{\omega_0}^+$ and $\chi_{\omega_0}^-$.

Remark 4.2.8. As before, since $-R_{T^s}^G(\omega) = -R_{T^s}^G(\omega^q)$ in the Gl_2 case, we are going to have $-R_{T^s}^G(\omega) = -R_{T^s}^G(\omega^{-1})$ in the Sl_2 case, so we only have $\frac{q-1}{2}$ of this type ($\omega^2 \neq 1$).

Hence, the character table of $\text{Sl}_2(q)$ looks like (see [4]):

Table 4.2: Character Table of $\text{Sl}_2(q)$.

| classes | $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $a = \pm 1$ | $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ $a = \pm 1$ | $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ $a \notin \{1, -1\}$ | $\begin{pmatrix} x & 0 \\ 0 & x^q \end{pmatrix}$ $x \neq x^q$ |
|-----------------------|---|---|---|--|
| # of classes | 2 | 4 | $(q-3)/2$ | $(q-1)/2$ |
| class size | 1 | $(q^2-1)/2$ | $q(q+1)$ | $q(q-1)$ |
| $R_T^G(\alpha)$ | $(q+1)\alpha(a)$ | $\alpha(a)$ | $\alpha(a) + \alpha(a^{-1})$ | 0 |
| $\chi_{\alpha_0}^\pm$ | $\frac{(q+1)}{2}\alpha_0(a)$ | $\frac{\alpha_0(a)}{2}(1 \pm \varphi)$ | $\alpha_0(a)$ | 0 |
| $-R_{T^s}^G(\omega)$ | $(q-1)\omega(a)$ | $-\omega(a)$ | 0 | $-(\omega(x) + \omega(x^q))$ |
| $\chi_{\omega_0}^\pm$ | $\frac{(q-1)}{2}\omega_0(a)$ | $\frac{\omega_0(a)}{2}(-1 \pm \varphi)$ | 0 | $-\omega_0(x)$ |
| 1 | 1 | 1 | 1 | 1 |
| St_G | q | 0 | 1 | -1 |

where b is either 1 or y for some fixed non square $y \in \mathbb{F}_q \setminus (\mathbb{F}_q)^2$, $a \in \mathbb{F}_q^\times$ and $x \in \mathbb{F}_{q^2}^\times$ is a norm 1 element of degree 2 (in other words: $x^{q+1} = 1$ and $x \neq \pm 1$), and in the $-R_{T^s}^G(\omega)$ row, the ω is taken in μ_{q+1} with $\omega^2 \neq 1$.

The term φ is a short for $\alpha_0(ab)\sqrt{\alpha_0(-1)q}$ and for our purposes is not really important since we only want the central values of the characters (i.e.: namely those of the first row).

4.3 The number of \mathbb{F}_q -points of $\mathcal{M}^g(\text{Sl}_2)$

Thanks to formula (2.12) we can compute the number $\tilde{N}_2^g(q) = N_2^g(q)|H|$ of solutions of

$$[y_1, x_1] \dots [y_g, x_g] = z$$

for $x_i, y_i \in \text{Sl}_2(q)$ and $z = -Id$ as:

$$\tilde{N}_2^g(q) = \sum_{\chi \in \text{Irr}(H)} \chi(z) \left(\frac{|H|}{\chi(1)} \right)^{2g-1} \quad (4.1)$$

where $H = \text{Sl}_2(q)$.

By plugging the values from the first column of table 4.2 in (4.1), and regarding z as $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for $a = -1$, the character sum (4.1) becomes:

$$\begin{aligned} \tilde{N}_2^g(q) &= \sum_{i=1}^{\frac{q-3}{2}} (q+1)(-1)^i \left(\frac{|H|}{q+1} \right)^{2g-1} \\ &\quad + 2 \frac{q+1}{2} (-1)^{\frac{q-1}{2}} \left(\frac{|H|}{(q+1)/2} \right)^{2g-1} \\ &\quad + \sum_{i=1}^{\frac{q-1}{2}} (q-1)(-1)^i \left(\frac{|H|}{q-1} \right)^{2g-1} \\ &\quad + 2 \frac{q-1}{2} (-1)^{\frac{q+1}{2}} \left(\frac{|H|}{(q-1)/2} \right)^{2g-1} \\ &\quad + 1 \left(\frac{|H|}{1} \right)^{2g-1} \\ &\quad + q \left(\frac{|H|}{q} \right)^{2g-1} \end{aligned} \quad (4.2)$$

where different lines match the corresponding rows in the character table, for instance the second line is

$$\chi_{\alpha_0}^+(z) \left(\frac{|H|}{\chi_{\alpha_0}^+(1)} \right)^{2g-1} + \chi_{\alpha_0}^-(z) \left(\frac{|H|}{\chi_{\alpha_0}^-(1)} \right)^{2g-1}.$$

Dividing equation (4.2) by $|H|$ and simplifying the sums we get

$$\begin{aligned}
N_2^g(q) = & -(1 + (-1) + \dots + (-1)^{\frac{q-3}{2}-1}) \left(\frac{|H|}{q+1} \right)^{2(g-1)} \\
& + (-1)^{\frac{q-1}{2}} 2^{2g-1} \left(\frac{|H|}{q+1} \right)^{2(g-1)} \\
& - (1 + (-1) + \dots + (-1)^{\frac{q-1}{2}-1}) \left(\frac{|H|}{q-1} \right)^{2(g-1)} \\
& + (-1)^{\frac{q+1}{2}} 2^{2g-1} \left(\frac{|H|}{q-1} \right)^{2(g-1)} \\
& + \left(\frac{|H|}{1} \right)^{2(g-1)} + \left(\frac{|H|}{q} \right)^{2(g-1)}
\end{aligned} \tag{4.3}$$

which in turn reduces to:

$$\begin{aligned}
N_2^g(q) = & - \left(\frac{1 + (-1)^{\frac{q-1}{2}}}{2} \right) \left(\frac{|H|}{q+1} \right)^{2(g-1)} \\
& + (-1)^{\frac{q-1}{2}} 2^{2g-1} \left(\frac{|H|}{q+1} \right)^{2(g-1)} \\
& - \left(\frac{1 + (-1)^{\frac{q+1}{2}}}{2} \right) \left(\frac{|H|}{q-1} \right)^{2(g-1)} \\
& + (-1)^{\frac{q+1}{2}} 2^{2g-1} \left(\frac{|H|}{q-1} \right)^{2(g-1)} \\
& + \left(\frac{|H|}{1} \right)^{2(g-1)} + \left(\frac{|H|}{q} \right)^{2(g-1)}
\end{aligned} \tag{4.4}$$

and finally to the quasi-polynomial:

$$\begin{aligned}
N_2^g(q) = & \left((-1)^{\frac{q-1}{2}} 2^{2g-1} - \left(\frac{1 + (-1)^{\frac{q-1}{2}}}{2} \right) \right) \left(\frac{|H|}{q+1} \right)^{2g-2} \\
& + \left((-1)^{\frac{q+1}{2}} 2^{2g-1} - \left(\frac{1 + (-1)^{\frac{q+1}{2}}}{2} \right) \right) \left(\frac{|H|}{q-1} \right)^{2g-2} \\
& + \left(\frac{|H|}{1} \right)^{2g-2} + \left(\frac{|H|}{q} \right)^{2g-2}
\end{aligned} \tag{4.5}$$

which, for $q \equiv 1 \pmod{4}$ becomes:

$$\begin{aligned} N_2^g(q) = & (2^{2g-1} - 1) \left(\frac{|H|}{q+1} \right)^{2g-2} - 2^{2g-1} \left(\frac{|H|}{q-1} \right)^{2g-2} \\ & + \left(\frac{|H|}{1} \right)^{2g-2} + \left(\frac{|H|}{q} \right)^{2g-2}. \end{aligned} \quad (4.6)$$

Since $|H| = q(q-1)(q+1)$ the number of points $N_2^g(q)$ depends polynomially in q (provided $q \equiv 1 \pmod{4}$), with that polynomial having integer coefficients.

4.4 Remarks

From the formulas obtained in last section we can deduce the following facts about $\mathcal{M}^g(\mathrm{Sl}_2)$ and their E -polynomials:

1. Formula (4.6) implies that the Character Varieties $\mathcal{M}^g(\mathrm{Sl}_2)$ have polynomial count and thanks to Theorem 3.3.1, the polynomial expression found for N_2^g are the E -polynomials $E(q; \mathcal{M}^g(\mathrm{Sl}_2))$ of $\mathcal{M}^g(\mathrm{Sl}_2)$.
2. For $g = 1$, the quasi-polynomial computed in (4.5) tells us the number of pairs of matrices (up to PGL_2 -conjugation) in $\mathrm{Sl}_2(q)$ whose commutator is $-Id$ is 1 because

$$\begin{aligned} N_2^g(q) = & (-1)^{\frac{q-1}{2}} 2 - \left(\frac{1 + (-1)^{\frac{q-1}{2}}}{2} \right) \\ & + (-1)^{\frac{q+1}{2}} 2 - \left(\frac{1 + (-1)^{\frac{q+1}{2}}}{2} \right) \\ & + 1 + 1 \end{aligned}$$

and since $\left((-1)^{\frac{q-1}{2}} + (-1)^{\frac{q+1}{2}}\right) = 0$,

$$\begin{aligned} N_2^g(q) &= \left((-1)^{\frac{q-1}{2}} + (-1)^{\frac{q+1}{2}}\right) 2 \\ &\quad - \left(\frac{2 + (-1)^{\frac{q-1}{2}} + (-1)^{\frac{q+1}{2}}}{2}\right) + 2 \\ N_2^g(q) &= 1. \end{aligned}$$

Therefore, the equation $[Y, X] = -Id$ has a unique solution in Sl_2 (modulo PGL_2 -action).

3. For $g \geq 2$ we see that the principal coefficient of the E -polynomial found in (4.6) comes from the term $\left(\frac{|H|}{1}\right)^{2g-2}$ and it is 1. This together with the conclusion of last remark (for the case $g = 1$) proves that the counting functions of the $\mathcal{M}^g(\text{Sl}_2)$ are monic of degree $\deg_q P = 3(2g - 2)$. Then, the character varieties $\mathcal{M}^g(\text{Sl}_2)$ are connected of dimension $\dim(X)$. This and the next remark are particular cases of Corollary 1.0.3.
4. These E -polynomials are easily seen to be palindromic (i.e.: they remain unchanged when their coefficients are reversed). Being palindromic for a polynomial $P \in \mathbb{Q}[q]$ means $P(q) = q^{\deg P} P(q^{-1})$. In our case, the polynomials N_2^g have degrees $3(2g - 2)$ and verify:

$$\begin{aligned} N_2^g(q^{-1}) &= (2^{2g-1} - 1) (q^{-1}(q^{-1} - 1))^{2g-2} - 2^{2g-1} (q^{-1}(q^{-1} + 1))^{2g-2} \\ &\quad + (q^{-1}(q^{-1} + 1)(q^{-1} - 1))^{2g-2} + ((q^{-1} + 1)(q^{-1} - 1))^{2g-2} \end{aligned}$$

multiplying both sides by q^{6g-6}

$$\begin{aligned}
q^{6g-6} N_2^g(q^{-1}) &= (2^{2g-1} - 1) (q^{-1}(q^{-1} - 1)q^3)^{2g-2} - 2^{2g-1} (q^{-1}(q^{-1} + 1)q^3)^{2g-2} \\
&\quad + (q^{-1}(q^{-1} + 1)(q^{-1} - 1)q^3)^{2g-2} + ((q^{-1} + 1)(q^{-1} - 1)q^3)^{2g-2} \\
q^{6g-6} N_2^g(q^{-1}) &= (2^{2g-1} - 1) (q(q - 1))^{2g-2} - 2^{2g-1} (q(q + 1))^{2g-2} \\
&\quad + ((q + 1)(q - 1))^{2g-2} + (q(q + 1)(q - 1))^{2g-2}
\end{aligned}$$

whose right-hand side is $N_2^g(q)$.

5. The Euler Characteristic of $\mathcal{M}^g(\text{Sl}_2)$ is calculated evaluating at $q = 1$ the polynomial $N_2^g(q) = E(q; \mathcal{M}^g(\text{Sl}_2))$ from (4.6), which in this case is

$$\begin{aligned}
N_2^g(1) &= (2^{2g-1} - 1) (q(q - 1))^{2g-2} \Big|_{q=1} - 2^{2g-1} (q(q + 1))^{2g-2} \Big|_{q=1} \\
&\quad + (q(q - 1)(q + 1))^{2g-2} \Big|_{q=1} + ((q - 1)(q + 1))^{2g-2} \Big|_{q=1} \\
&= -2^{4g-3}.
\end{aligned}$$

We point out that the only term that survives (when evaluated at $q = 1$) is the one corresponding to the characters of type τ_2 that split when restricted to $\text{Sl}_2(q)$.

Chapter 5

E-polynomial of $\mathcal{M}^g(\mathrm{Sl}_n)$

In this chapter we outline the computation of $E(q; \mathcal{M}^g(\mathrm{Sl}_n))$, the *E*-polynomials of the twisted character varieties $\mathcal{M}^g(\mathrm{Sl}_n)$, by finding the counting functions $N_n^g(q)$ and proving they are polynomials in q for those prime powers q satisfying (1.9). Those polynomials turn out to be the sought *E*-polynomials thanks to Theorem 3.3.1.

The strategy to calculate $N_n^g(q)$ is the following

1. Write the character sum from (2.12) as a sum in terms of $\mathrm{Irr}(\mathrm{Gl}_n(q))$.
2. Gather together the summands with the same type.
3. In last summation, separate the terms by splittings of the $\chi \in \mathrm{Irr}(\mathrm{Gl}_n(q))$ when restricted to $\mathrm{Sl}_n(q)$, according to their stabilizers.
4. Filter the summands by degree of the classes of the characters $\gamma \in \Gamma$ from 2.2.4.
5. Reduce the sum to one in terms of characters $\alpha \in \Gamma$.
6. Go all the way back to the original sum with the results obtained.

Throughout this chapter g and n are positive integers, q is a prime power with residue 1 modulo n , δ is a fixed generator of the cyclic group $\Gamma_1 = \widehat{\mathbb{F}}_q^\times = \langle \delta \rangle$, $\mathrm{Gl}_n(q)$ will be noted by G , $\mathrm{Sl}_n(q)$ will be noted by H , z is $\zeta_n Id$ (an order n central element in H), χ will note an irreducible character of G , θ an irreducible character of H , s and t will be sizes of orbits of associated characters verifying $st = q - 1$.

We leave many of the calculations for Chapter 6.

5.1 First Reduction: From Sl_n to Gl_n

In order to apply Katz's result (Theorem 3.3.1 here, or [9], appendix), we ought to compute $N_n^g(q)$, the number of \mathbb{F}_q -points of $\mathcal{M}^g(\mathrm{Sl}_n)$. We will see that under condition (1.9) this number depends polynomially in q .

By Corollary (3.1.3), Remark 3.4.3 and formula (2.12) the number $N_n^g(q)$ of points of $\mathcal{M}^g(\mathrm{Sl}_n)(q)$ is computed by the formula:

$$N_n^g(q) = \frac{1}{|H|} \sum_{\theta \in \mathrm{Irr}(H)} \left(\frac{|H|}{\theta(1)} \right)^{2g-1} \theta(z). \quad (5.1)$$

Thanks to the characterization of $\mathrm{Irr}(H)$ given in Section 2.2.3, we can express this summation in terms of characters from $\mathrm{Irr}(G)$.

According to Remark 2.2.11, there is a bijective correspondence between orbits under G/H -conjugation of $\mathrm{Irr}(H)$ and orbits under $(\widehat{G/H})$ -multiplication of $\mathrm{Irr}(G)$, and such correspondence was given by restricting to H a character $\chi \in \mathrm{Irr}(G)$ and taking its irreducible components $\theta \in \mathrm{Irr}(H)$.

Grouping together G/H -orbits of $\text{Irr}(H)$ in the sum (5.1) we get

$$\begin{aligned}
N_n^g(q) &= \frac{1}{|H|} \sum_{\theta \in \text{Irr}(H)} \left(\frac{|H|}{\theta(1)} \right)^{2g-1} \theta(z) \\
&= \frac{1}{|H|} \sum_{\theta \in \text{Irr}(H)} \left(\frac{|H|}{\theta(1)} \right)^{2g-1} \theta(z) \\
&= \frac{1}{|H|} \sum_{[\theta] \in \text{Irr}(H)/(G/H)} \left(\frac{|H|}{[\theta](1)} \right)^{2g-1} \sum_{\theta \in [\theta]} \theta(z) \quad (5.2)
\end{aligned}$$

where $[\theta] \subseteq \text{Irr}(H)$ stands for a G/H -orbit and $[\theta](1)$ is the degree of any element in such orbit.

Passing to the $\text{Irr}(G)$ side by means of the correspondence from Remark 2.2.11, and using that:

- $\sum_{\theta \in [\theta]} \theta(z) = \chi(z)$ for $\chi \in \text{Irr}(G)$ the corresponding character whose H restriction affords $[\theta]$,
- The degree $\chi(1)$ is $t \theta(1)$, where

$$t = t(\chi) := |[\theta]|, \quad (5.3)$$

we end up with:

$$N_n^g(q) = \frac{1}{|H|} \sum_{[\chi]/\chi \in \text{Irr}(G)} \left(\frac{|H|}{\chi(1)} \right)^{2g-1} t(\chi)^{2g-1} \chi(z) \quad (5.4)$$

where $[\chi]$ is the orbit of $\chi \in \text{Irr}(G)$ of some size $s = s(\chi) := |[\chi]|$, t is the size of the G/H -orbit $[\theta]$ corresponding to the constituents of the restriction χ_H and their sizes satisfy $st = q - 1$ (since $G/H = \mathbb{F}_q^\times$ is cyclic of order $q - 1$).

Remark 5.1.1. A character $\chi \in \text{Irr}(G)$ whose $(\widehat{G/H})$ -orbit has size s must necessarily have a stabilizer of order $t = (q - 1)/s$. Since $(\widehat{G/H}) = \widehat{\mathbb{F}_q^\times} = \Gamma_1$ is cyclic, such stabilizer is generated by δ^s (Remark 2.2.6). By Remark 2.2.24 we know δ^s acts on the support of the Θ -partitions $\Lambda : \Gamma \rightarrow \mathcal{P}$ associated to χ . Since multiplication is a free action, by Theorem 2.2.4 we conclude that t divides each product $dm_{d,\lambda}$, and hence divides n .

5.2 Second Reduction: From Gl_n to types

We gather together from (5.4) those characters $\chi \in \text{Irr}(G)$ having the same type τ .

Using formula (2.27), equation (5.4) becomes

$$\begin{aligned}
N_n^g(q) &= \frac{1}{|H|} \sum_{\substack{[\chi], t \\ \chi \in \text{Irr}(G) \\ t=t(\chi)}} \left(\frac{|H|}{\chi(1)} \right)^{2g-1} t^{2g-1} \chi(z) \\
&= \sum_{\substack{[\chi], t \\ \chi \in \text{Irr}(G) \\ t=t(\chi)}} \left(\frac{|H|}{\chi(1)} \right)^{2(g-1)} t^{2g-1} \Delta_\chi(\zeta_n) \\
&= \sum_{\substack{\tau, t \\ |\tau|=n \\ t|n}} \left(\frac{|H|}{\chi_\tau(1)} \right)^{2(g-1)} t^{2g-1} \sum_{\substack{[\chi] \in \text{Irr}(G)/\widehat{\mathbb{F}_q^\times} \\ t(\chi)=t \\ \tau(\chi)=\tau}} \Delta_\chi(\zeta_n) \\
&= \sum_{\substack{\tau, t \\ |\tau|=n \\ t|n}} \left(\frac{|H|}{\chi_\tau(1)} \right)^{2(g-1)} t^{2g-1} C_\tau^t \tag{5.5}
\end{aligned}$$

where

$$C_\tau^t := \sum_{\substack{[\chi] \in \text{Irr}(G)/\widehat{\mathbb{F}_q^\times} \\ |\chi| = s \\ \tau(\chi) = \tau}} \Delta_\chi(\zeta_n) \quad (5.6)$$

and $\Delta_\chi(\zeta_n)$ is the $\Delta_\Lambda(\zeta_n) = \prod_\gamma \gamma^{|\Lambda(\gamma)|}(\zeta_n)$ (evaluated in Γ_1), from (2.27).

Using that $|H| = (q-1)^{-1}|G|$ and plugging in (2.30) in (5.5) we find

$$N_n^g(q) = \sum_{\tau, t} \left(q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} t^{2g-1} C_\tau^t. \quad (5.7)$$

Remark 5.2.1. In Section 5.6 we will see that C_τ^t does not depend on q (at least as soon as q satisfies (1.9)) and hence it is just a number. This, together with the fact that the factors $\left(\frac{|H|}{\chi_\tau(1)} \right) = \left(q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{q-1} \right)$ are polynomials in $\mathbb{Z}[q]$ (formulas (2.1), (2.2)) and Remark 2.2.20), will lead us to the remarkable fact that $N_n^g(q)$ is a polynomial for those values of q . Then, by Remark 3.4.1, the Character Variety $\mathcal{M}^g(\text{Sl}_n)$ results polynomial count.

Remark 5.2.2. For $g \geq 2$ the highest degree term in (5.7) is the one corresponding to the type τ_p with unique nonzero multiplicity $(\tau_p)_{1,(1,\dots,1)} = 1$, namely that of 1_G , the trivial character¹ (see Remarks 2.2.18 and 2.2.19).

Remark 5.2.3. Since the $\widehat{(G/H)}$ -orbit of 1_G is the set of all τ_p -type characters in $\text{Irr}(G)$, and has size $s = q-1$, the corresponding G/H -orbit will have size $t = 1$ (Remark 2.2.11), hence their H -restriction are irreducible².

¹also: the biggest $\left(\frac{|H|}{\chi(1)} \right)$ is attained by the character with smallest degree $\chi(1)$.

²the restriction of the trivial character 1_G is 1_H , hence it cannot have more than one component.

Remark 5.2.4. The only t for which $C_{\tau_p}^t \neq 0$ is $t = 1$. Since the sum (5.6) giving $C_{\tau_p}^1$ has only one summand, whose value is 1, the principal coefficient of $N_n^g(q)$ will be 1 and its degree $(2g - 2)(n^2 - 1)$. This, together with Corollary 5.8.2 proves Corollary 1.0.3.

Remark 5.2.5. From (5.6) and the fact that $|\Lambda(\gamma)| = |\Lambda'(\gamma)|$ we see that C_τ^t and $C_{\tau'}^t$ have the same value.

5.3 Third Reduction: Controlling Stabilizers

5.3.1 Fixed Stabilizer

Let us focus now in the calculation of C_τ^t . Since they are defined in (5.6) by a sum on $(\widehat{G/H})$ -orbits of characters, all of which have size $s = (q - 1)/t$, we can transform such sum in one on characters by overcounting the character classes:

$$sC_\tau^t = \sum_{\substack{\chi \in \text{Irr}(G) \\ |[\chi] | = s \\ \tau(\chi) = \tau}} \Delta_\chi(\zeta_n) \quad (5.8)$$

where $(\widehat{G/H})$ -conjugate characters are considered separately. The others restrictions on the sum remain the same, in other words, we are summing over characters $\chi \in \text{Irr}(G)$ (instead of character classes) of type τ and whose stabilizers have order exactly t .

Remark 5.3.1. For a $\chi \in \text{Irr}(G)$ to have stabilizer of order t is the same as verifying $\chi = \delta^s \chi$ and s the smaller such power of δ that stabilizes it.

5.3.2 Bounded Stabilizer

It will be convenient to work with characters that are stable by a fixed power of δ , regardless of the actual size of its stabilizer. And that amounts to take a cumulative sum of these sC_τ^t which we call:

$$\widehat{S}(t, \tau) := \sum_{\substack{k|\frac{n}{t} \\ n=|\tau|}} \frac{s}{k} C_\tau^{tk} = \sum_{\substack{\chi \in \text{Irr}(G) \\ \delta^s \chi = \chi \\ \tau(\chi) = \tau}} \Delta_\chi(\zeta_n). \quad (5.9)$$

Remark 5.3.2. Thanks to the Möbius Inversion Formula, we can solve for C_τ^t in (5.9) getting

$$C_\tau^t = \frac{t}{q-1} \sum_{\substack{k|\frac{n}{t} \\ n=|\tau|}} \mu(k) \widehat{S}(kt, \tau). \quad (5.10)$$

Remark 5.3.3. Plugging $t = 1$ in (5.9), and keeping in mind that δ^s becomes trivial in this case, we get

$$\widehat{S}(1, \tau) = \sum_{\substack{\chi \in \text{Irr}(G) \\ \tau(\chi) = \tau}} \Delta_\chi(\zeta_n) = C_\tau$$

were the coefficients C_τ are those from [9] (3.2.2).

5.4 Fourth Reduction: The New-degree

In order to evaluate the Δ_χ 's, we need to consider one extra feature we shall define next.

Definition 5.4.1. The Frobenius automorphism on Γ induces an action in the quotient $Frob_q \curvearrowright \Gamma / \langle \delta^s \rangle$. Define the *newdegree* of $\alpha \in \Gamma$ relative to t as

the degree of its class $[\alpha] \in \Gamma / \langle \delta^s \rangle$, namely

$$\deg^t(\alpha) := \deg([\alpha]) = \left| \left\{ [\alpha], [\alpha]^q, [\alpha]^{q^2}, \dots \right\} \right|.$$

Remark 5.4.1. Given $t|n$ and α of degree d and newdegree \widehat{d} we have

$$\widehat{dt} = |(\langle \text{Frob}_q \rangle \times \langle \delta^s \rangle) \alpha| = \left| \left\{ \delta^{si} \alpha^{q^j} \right\}_{i,j} \right| = d\widehat{t}$$

for some integer \widehat{t} .

Remark 5.4.2. Given a $\chi \in \text{Irr}(G)$ stable by multiplication by δ^s and Λ the corresponding multipartition, we have that the product \widehat{dt} from last remark divides $dm_{d,\lambda}(\Lambda)$ for any $\lambda \in \mathcal{P}$, since for a fixed $\alpha \in \Gamma$ every i and j , the partitions $\Lambda(\delta^{si} \alpha^{q^j})$ are the same.

Remark 5.4.3. As a particular case of Lemma 1.0.4 from Appendix 1 we get that \widehat{d} divides d , and hence, by last remark \widehat{t} divides t . This follows by realizing that d is the size of a Frob_q -orbit of a character, and that \widehat{d} is the size of a Frob_q -orbit of character classes.

Definition 5.4.2. We say that a type τ is pure of degree d if all of its multiplicities $(\tau)_{d',\lambda}$ are 0 for $d' \neq d$. This is the same as saying that all the $\gamma \in \Gamma$ in the support of a Λ of type τ have degree d .

We also say a character χ is pure if its type is so.

Definition 5.4.3. Let us consider t some divisor of n and χ a character of G corresponding to a multi-partition $\Lambda \in \mathcal{P}_n^\Theta$. Let us further assume χ stable by

δ^{s3} . We say χ is pure of newdegree \widehat{d} if all the γ 's in the support of Λ have newdegree \widehat{d} .

Notation. In this case we write $\deg^t(\chi) = \widehat{d}$.

Remark 5.4.4. We claim $\widehat{S}(t, \tau) = 0$ unless τ is pure of some degree d , in particular $d|n$. This will be proven in Lemma 6.2.1 of next Chapter.

Writing

$$S(t, \widehat{d}, \tau) := \sum_{\substack{\chi \in \text{Irr}(G) \\ \delta^s \chi = \chi \\ \deg^t(\chi) = \widehat{d} \\ \tau(\chi) = \tau}} \Delta_\chi(\zeta_n) \quad (5.11)$$

we get

$$\widehat{S}(t, \tau) = \sum_{\widehat{d}|d} S(t, \widehat{d}, \tau) \quad (5.12)$$

where d is the degree of all the characters in the support of τ . Here we are implicitly using the fact that the characters of mixed newdegrees add up to zero, which will be proved in Lemma 6.2.2.

5.5 Fifth Reduction: Writing it in terms of Γ

5.5.1 Ordering the characters

Now take some character χ corresponding to some function $\Lambda \in \mathcal{P}_n^\Theta$, pure of degree d and of newdegree \widehat{d} , for some given t divisor of n .

³remember $st = q - 1$

Recalling (2.29)

$$\Delta_\Lambda(\zeta_n) = \prod_{\{\gamma\} \in \Theta} \gamma^{|\Lambda(\gamma)|}(\zeta_n)$$

the product being taken over $Frob_q$ -orbits, and the evaluations are done in Γ_d .

Writing it in terms of $\widehat{\mathbb{F}}_q^\times$ -orbits we have

$$\begin{aligned} \Delta_\Lambda(\zeta_n) &= \prod_{\{\gamma\} \in \Gamma / \langle Frob_q, \delta^s \rangle} \prod_{i=0}^{\widehat{t}-1} (\delta^{si} \gamma)^{|\Lambda(\gamma)|}(\zeta_n) \\ &= \prod_{\{\gamma\} \in \Gamma / \langle Frob_q, \delta^s \rangle} \left(\gamma^{\widehat{t}|\Lambda(\gamma)|}(\zeta_n) \prod_{i=0}^{\widehat{t}-1} (\delta^{dsi})^{|\Lambda(\gamma)|}(\zeta_n) \right) \end{aligned}$$

where the δ is evaluated in Γ_1 , and that is why we add the factor d to the exponent. Grouping together and pulling out all the powers of δ we get

$$\begin{aligned} \Delta_\Lambda(\zeta_n) &= \prod_{\{\gamma\} \in \Gamma / \langle Frob_q, \delta^s \rangle} \gamma^{|\Lambda(\gamma)|}(\zeta_n^{\widehat{t}}) \delta(\zeta_n)^{ds|\Lambda(\gamma)| \sum_{i=0}^{\widehat{t}-1} i} \\ &= \prod_{\{\gamma\} \in \Gamma / \langle Frob_q, \delta^s \rangle} \gamma^{|\Lambda(\gamma)|}(\zeta_n^{\widehat{t}}) \delta(\zeta_n)^{ds|\Lambda(\gamma)| \binom{\widehat{t}}{2}} \\ &= \delta(\zeta_n)^{ds \binom{\widehat{t}}{2} \sum |\Lambda(\gamma)|} \prod_{\{\gamma\} \in \Gamma / \langle Frob_q, \delta^s \rangle} \gamma^{|\Lambda(\gamma)|}(\zeta_n^{\widehat{t}}) \quad (5.13) \end{aligned}$$

where the summation $ds \binom{\widehat{t}}{2} \sum |\Lambda(\gamma)|$ in the first exponent has the same indexing as the product, namely the $Frob_q$ -orbits of the classes of characters $[\gamma] \in \Gamma / \langle \delta^s \rangle$. The δ is evaluated as in Γ_1 and the γ 's are evaluated as in Γ_d .

We next overcount every χ in the summation (5.11) by ordering the γ 's in tuples of character classes $[\alpha_i] \in \Gamma / \langle \delta^s \rangle$ and distinguishing between

$Frob_q$ -conjugates as well and then we express:

$$S(t, \widehat{d}, \tau) = \frac{\delta(\zeta_n)^{ds \binom{t}{2} \sum \lambda_i}}{\widehat{d}^m \prod \widehat{m}_\lambda!} Z(d, t, \widehat{d}, \vec{\lambda}) \quad (5.14)$$

where \widehat{m}_λ is defined by $\widehat{d}t\widehat{m}_\lambda = m_{d,\lambda}d$, $\vec{\lambda} = (\lambda_1, \dots, \lambda_m)$ is the list of the exponents corresponding to a fixed ordering of the $|\Lambda(\gamma)|$'s, and

$$Z(d, t, \widehat{d}, \vec{\lambda}) := \sum_{\substack{([\alpha_1], \dots, [\alpha_m]) \in (\Gamma_d / \langle \delta^s \rangle)^m \\ \{[\alpha_i]\} \neq \{[\alpha_j]\}, i \neq j \\ \deg(\alpha_i) = d \\ \deg([\alpha_i]) = \widehat{d}}} \prod_{i=1}^m \alpha_i^{\lambda_i}(\zeta_n^t) \quad (5.15)$$

where the α_i 's are evaluated as in Γ_d .

Remark 5.5.1. The range of the sum should be read as follows:

- We are taking m -tuples of classes of characters $[\alpha]$ where $\alpha \in \Gamma_d$. We identify two of them whenever their entries differ only by factors of δ^s , namely if they are δ^s -conjugate.
- Different entries in the m -tuple cannot be $Frob_q$ -conjugates. In particular they correspond to different $\alpha_i \in \Gamma_d$.
- All the entries have degree exactly d .
- All the entries have newdegree exactly \widehat{d} relative to t . That is to say, the degree of the class $[\alpha_i] \in \Gamma / \langle \delta^s \rangle$ is \widehat{d} .

Remark 5.5.2. The factor $\delta(\zeta_n)^{ds \binom{t}{2} \sum |\Lambda(\gamma)|}$ is ± 1 and it is 1 under condition (1.9). This will say that for those prime powers $q \equiv 1 \pmod{n}$ (i.e.: those for

which $\mathcal{M}^g(\mathrm{Sl}_n)$ is defined) the function $N_n^g(q)$ is a quasi-polynomial in q , with period at most 2.

Remark 5.5.3. Since the evaluations $\alpha_i(\zeta_n^{\widehat{t}})$ are done in Γ_d , they are well defined in $\Gamma/\langle\delta^s\rangle$ because $\zeta_n^{\widehat{t}} \in \mathrm{Ker}(^T N_1^d(\delta^s))$ (see Remark 2.2.5). This last assertion holds since

$$^T N_1^d(\delta^s)(\zeta_n^{\widehat{t}}) = \delta^s(N_1^d(\zeta_n^{\widehat{t}})) = \delta^s(\zeta_n^{d\widehat{t}}) = \delta^{sd\widehat{t}}(\zeta_n)$$

evaluated as in Γ_1 , which is 1 as $sd\widehat{t} = st\widehat{d} = (q-1)\widehat{d}$ is divisible by n .

5.5.2 Ignoring degrees

Now define the accumulated sums of these $Z(d, t, \widehat{d}, \vec{\lambda})$ in order to get rid of the condition on the degrees of the characters α_i . Note that this will not remove the condition on their newdegrees.

Definition 5.5.1. Keeping in mind that the degree of a character must be divisible by its newdegree \widehat{d} (Remark 5.4.3) we define:

$$\widehat{Z}(D, t, \widehat{d}, \vec{\lambda}) := \sum_{\substack{d|D \\ \widehat{d}|d}} Z(d, t, \widehat{d}, \vec{\lambda}). \quad (5.16)$$

Remark 5.5.4. Then, inverting (5.16) with formula (2.7) we will have:

$$Z(D, t, \widehat{d}, \vec{\lambda}) = \sum_{\substack{d|D \\ \widehat{d}|d}} \mu\left(\frac{D}{d}\right) \widehat{Z}(d, t, \widehat{d}, \vec{\lambda}). \quad (5.17)$$

Remark 5.5.5. We need to point out that each term $Z(d, t, \widehat{d}, \vec{\lambda})$ corresponds to a sum over m -tuples of classes of characters of Γ_d where the d changes, and

hence so does the exponent \widehat{t} , being \widehat{dt}/d . These two changes compensate each other in the sense that for a given $\alpha \in \Gamma_d$ evaluating at $\zeta_n^{\widehat{dt}/d}$ regarding α as in Γ_d and evaluating at $\zeta_n^{\widehat{dt}/D}$ regarding α as in Γ_D yields to the same result since in the second case one needs to compose with N_d^D which amounts to multiplying the exponent of ζ_n by a factor of D/d .

From this remark we conclude that the sum defining $\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda})$ could be computed with a fixed $\widehat{t} = \frac{t\widehat{d}}{D}$ and all the evaluations with the α 's regarded as in Γ_D even when the α has a smaller degree.

Thus, we could compute \widehat{Z} as:

$$\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda}) = \sum_{\substack{([\alpha_1], \dots, [\alpha_m]) \in (\Gamma_D / \langle \delta^s \rangle)^m \\ \{[\alpha_i]\} \neq \{[\alpha_j]\}, i \neq j \\ \deg([\alpha_i]) = \widehat{d}}} \prod_{i=1}^m \alpha_i^{\lambda_i}(\zeta_n^{\widehat{t}}). \quad (5.18)$$

Remark 5.5.6. Note that in 5.18 there are summands $\prod_{i=1}^m \alpha_i^{\lambda_i}(\zeta_n^{\widehat{t}})$ not all of whose factors α_i have the same degree. It does not affect the value of the sum, because all this terms with mixed degrees will just cancel each other (Lemma 6.2.3)

5.6 Wrapping up

In section 6.3 we compute the values of the $\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda})$ and get the:

Lemma 5.6.1. *Given D , \widehat{d} and t divisors of n and a list of exponents $\overrightarrow{\lambda}$ verifying $D|t\widehat{d}$, $\widehat{d}|D$ and $n = \widehat{dt} \sum \lambda_i$, for q satisfying (1.9) we have:*

$$\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda}) = \begin{cases} \mu(\widehat{d})(-\widehat{d})^{m-1}(m-1)! \frac{q-1}{t} & \text{when } \gcd(D, t) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.19)$$

Remark 5.6.1. Since $\widehat{Z}(D, t, \widehat{d}, \vec{\lambda})$ is zero unless the three conditions $D|t\widehat{d}$, $\widehat{d}|D$ and $\gcd(D, t) = 1$ hold, the only nonzero values of \widehat{Z} arise when $\widehat{d} = D$.

Thus, under the condition (1.9) for q , and in view of last remark, formula (5.17) becomes:

$$\begin{aligned} Z(D, t, \widehat{d}, \vec{\lambda}) &= \mu\left(\frac{D}{\widehat{d}}\right) \widehat{Z}(\widehat{d}, t, \widehat{d}, \vec{\lambda}) \\ &= \mu\left(\frac{D}{\widehat{d}}\right) \mu(\widehat{d})(-\widehat{d})^{m-1}(m-1)!\frac{q-1}{t} \end{aligned} \quad (5.20)$$

provided $(\widehat{d}; t) = 1$, and zero otherwise.

Then, thanks to (5.20) and Remark 5.5.2, formula (5.14) becomes:

$$S(t, \widehat{d}, \tau) = \begin{cases} \mu\left(\frac{d}{\widehat{d}}\right)\frac{q-1}{t}(-1)^{m-1}\frac{\mu(\widehat{d})}{\widehat{d}}\frac{(m-1)!}{\prod \widehat{m}_\lambda!} & \text{when } \gcd(\widehat{d}, t) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.21)$$

where d is the degree of τ .

We recognize the factor

$$(-1)^{m-1}\frac{\mu(\widehat{d})}{\widehat{d}}\frac{(m-1)!}{\prod \widehat{m}_\lambda!} \quad (5.22)$$

as $C_{\widehat{\tau}}/(q-1) = C_{\widehat{\tau}}^0$ (calculated in [9] (3.4.2), see (5.25) in Remark 5.6.3 below) for some other type $\widehat{\tau}$, depending on τ , t and \widehat{d} .

Let's describe this new "quotient" type.

Definition 5.6.1. The type τ has only characters of degree d , and we are dividing them by a factor t_d to get \widehat{d} . We are also dividing the multiplicities $m_{d,\lambda}$ in order to get the \widehat{m}_λ in such a way to have

$$\begin{aligned} dm_{d,\lambda} &= \widehat{d}t\widehat{m}_\lambda \\ \frac{t_d}{t}m_{d,\lambda} &= \widehat{m}_\lambda. \end{aligned}$$

Remark 5.6.2. Writing t_m for the factor $\frac{t}{t_d}$, we verify

- $t = t_d t_m$,
- the size of $\widehat{\tau}$ is $|\widehat{\tau}| = \frac{n}{t}$,
- its degree $\deg(\widehat{\tau})$ is $\widehat{d} = \frac{d}{t_d}$ and
- the multiplicities $(\widehat{\tau})_{\widehat{d}, \lambda}$ are $\widehat{m}_\lambda = \frac{(\tau)_{d, \lambda}}{t_m}$.

Notation. We write $\tau/(t_d, t_m)$ for this new type $\widehat{\tau}$.

Formula (5.21) then becomes

$$S(t, \widehat{d}, \tau) = \begin{cases} \frac{q-1}{t} \mu(t_d) C_{\tau/(t_d, t_m)}^0 & \text{when } \gcd(\widehat{d}, t) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (5.23)$$

And for τ pure of degree d , and t a divisor of n , plugging (5.12) in (5.10), the coefficient C_τ^t becomes

$$\begin{aligned} C_\tau^t &= \frac{t}{q-1} \sum_{k|\frac{n}{t}} \mu(k) \widehat{S}(kt, \tau) \\ &= \frac{t}{q-1} \sum_{k|\frac{n}{t}} \mu(k) \sum_{\widehat{d}|d} S(kt, \widehat{d}, \tau) \end{aligned}$$

which, thanks to (5.23), boils down to

$$\begin{aligned} C_\tau^t &= \frac{t}{q-1} \sum_{k|\frac{n}{t}} \mu(k) \sum_{\substack{t_d, t_m \\ t_d t_m = tk \\ \exists \tau/(t_d, t_m) \\ (\widehat{d}, tk)=1}} \frac{q-1}{tk} \mu(t_d) C_{\tau/(t_d, t_m)}^0 \\ &= \sum_{\substack{t_d, t_m, k \\ \exists \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1 \\ t_d t_m = tk}} \frac{\mu(k)}{k} \mu(t_d) C_{\tau/(t_d, t_m)}^0 \end{aligned} \quad (5.24)$$

where this last sum is taken over all quotient types $\tau/(t_d, t_m)$ of degree \widehat{d} , such that $t_d t_m$ is multiple of t and is relatively prime to \widehat{d} .

Remark 5.6.3. From 5.3.3, (5.12) and (5.21) we recover formula (3.4.2) from [9],

$$C_\tau = \begin{cases} (-1)^{m-1} (q-1)^{\frac{\mu(d)}{d} \frac{(m-1)!}{\prod m_{d,\lambda}!}} & \text{for } \tau \text{ pure of degree } d \\ 0 & \text{otherwise} \end{cases} \quad (5.25)$$

5.7 Proof of Main Statements

We are now in conditions to prove Theorem 1.0.1.

Proof. The separated scheme from (3.4.1) gives us a spreading out of $\mathcal{M}^g(\mathrm{Sl}_n)$ thanks to (3.11).

By Remark 5.2.1 and (5.25), the counting functions $N_n^g(q)$ are polynomials in q .

This, together with (3.13) proves that the Character Varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ have polynomial count. Finally, by Katz's Theorem (3.3.1) the theorem follows. \square

As an immediate consequence we have Corollary 1.0.3 the proof of which we shall write now.

Proof. Plugging expression (5.24) in formula (5.7) and writing $\widehat{\tau}$ for the quo-

tient types $\tau/(t_d, t_m)$ we get:

$$\begin{aligned}
N_n^g(q) &= \sum_{\substack{\tau, t \\ t|n \\ |\tau|=n}} \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} t^{2g-1} C_\tau^t \\
&= \sum_{\substack{\tau, t \\ t|n \\ |\tau|=n}} \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} t^{2g-1} \sum_{\substack{t_d, t_m, k \\ \exists \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1 \\ t_d t_m = tk}} \frac{\mu(k)}{k} \mu(t_d) C_{\tau/(t_d, t_m)}^0 \\
&= \sum_{\substack{\tau, t \\ t|n \\ |\tau|=n}} \sum_{\substack{t_d, t_m, k, \widehat{\tau} \\ \widehat{\tau} = \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1 \\ t_d t_m = tk}} t^{2g} \frac{\mu(k)}{kt} \mu(t_d) C_{\widehat{\tau}}^0 \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)}
\end{aligned}$$

parameterizing the sum by pairs of types $\tau, \widehat{\tau}$ one of which is a quotient of the other

$$\begin{aligned}
N_n^g(q) &= \sum_{\substack{\tau, \widehat{\tau}, t_d, t_m \\ |\tau|=n \\ \widehat{\tau} = \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1}} \frac{1}{t_d t_m} \sum_{\substack{t, k \\ tk = t_d t_m}} t^{2g} \mu\left(\frac{t_d t_m}{t}\right) \mu(t_d) C_{\widehat{\tau}}^0 \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} \\
&= \sum_{\substack{\tau, \widehat{\tau}, t_d, t_m \\ |\tau|=n \\ \widehat{\tau} = \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1}} \frac{\mu(t_d)}{t_d t_m} C_{\widehat{\tau}}^0 \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{q-1} \right)^{2(g-1)} \sum_{\substack{t, k \\ tk = t_d t_m}} t^{2g} \mu\left(\frac{t_d t_m}{t}\right)
\end{aligned}$$

which, thanks to (2.9) gives:

$$N_n^g(q) = \frac{1}{(q-1)^{2(g-1)}} \sum_{\substack{\tau, \widehat{\tau}, t_d, t_m \\ |\tau|=n \\ \widehat{\tau} = \tau/(t_d, t_m) \\ (\widehat{d}, t_d t_m)=1}} \mu(t_d) \frac{O^{2g}(t_d t_m)}{t_d t_m} C_{\widehat{\tau}}^0 \left(q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q) \right)^{2(g-1)} \quad (5.26)$$

finishing thus the proof of formula (1.8). \square

5.8 Corollaries

From formula (5.26) of last section we can compute $N_n^g(1)$, which by Remark 3.3.3, computes the Euler Characteristic of $\mathcal{M}^g(\mathrm{Sl}_n)$.

Remark 5.8.1. As in 3.7 of [9], the only type τ whose $q^{\frac{n^2}{2}}\mathcal{H}_{\tau'}(q)$ has a simple zero at $q = 1$ is the one of degree $d = n$, and hence multiplicity $m_{n,(1)} = 1$ and 0 for other pairs $(d, \lambda) \neq (n, (1))$. Let us write τ_n for the unique type of degree n . All other types have at least a double zero at $q = 1$.

Remark 5.8.2. The value of

$$\frac{q^{\frac{n^2}{2}}\mathcal{H}_{\tau_n}(q)}{(q-1)}$$

at $q = 1$ is n .

Corollary 5.8.1. *For $g \geq 2$, the Euler Characteristic of $\mathcal{M}^g(\mathrm{Sl}_n)$ is $\mu(n)n^{4g-3}$.*

Proof. By Remark (3.3.3) we need to evaluate $N_n^g(1)$, which thanks to last two remarks gets simplified to

$$\begin{aligned} N_n^g(1) &= \sum_{\substack{\tau, \hat{\tau}, t_d, t_m \\ \hat{\tau} = \tau / (t_d, t_m) \\ (\hat{d}, t_d t_m) = 1}} \mu(t_d) \frac{O^{2g}(t_d t_m)}{t_d t_m} C_{\hat{\tau}}^0 \left(\frac{q^{\frac{n^2}{2}} \mathcal{H}_{\tau'}(q)}{(q-1)} \right)^{2(g-1)} \Big|_{q=1} \\ &= \sum_{\substack{\tau_n, \hat{\tau}_n, t_d, t_m \\ \hat{\tau}_n = \tau_n / (t_d, t_m) \\ (\hat{d}, t_d t_m) = 1}} \mu(t_d) \frac{O^{2g}(t_d t_m)}{t_d t_m} C_{\hat{\tau}_n}^0 n^{2(g-1)}. \end{aligned} \quad (5.27)$$

Since the only multiplicity of τ_n is 1, the only possible value for t_m is 1.

We shall now re-index the summation in (5.27) by $t = t_d$ a divisor of n .

And since the only possible quotient types are those with only one nonzero

multiplicity $m_{\widehat{d}(1)} = 1$, we can describe them by their degree, which is n/t for these cases.

Therefore, (5.27) becomes

$$\begin{aligned}
N_n^g(1) &= \sum_{\substack{t|n \\ (n/t, t)=1}} \mu(t) \frac{O^{2g}(t)}{t} C_{\widehat{\tau}_n}^0 n^{2(g-1)} \\
&= \sum_{\substack{t|n \\ (n/t, t)=1}} \mu(t) \frac{O^{2g}(t)}{t} \mu\left(\frac{n}{t}\right) \frac{t}{n} n^{2(g-1)} \\
&= \sum_{\substack{t|n \\ (n/t, t)=1}} \mu(t) \mu\left(\frac{n}{t}\right) O^{2g}(t) n^{2g-3} \tag{5.28}
\end{aligned}$$

and since $\mu(a)\mu(b) = \mu(ab)$ for relatively prime numbers a and b , we have

$$N_n^g(1) = \sum_{\substack{t|n \\ (n/t, t)=1}} \mu(n) O^{2g}(t) n^{2g-3}. \tag{5.29}$$

which, in turn, reduces to

$$N_n^g(1) = \mu(n) n^{2g-3} \sum_{t|n} O^{2g}(t), \tag{5.30}$$

since for square-free n , every divisor t is relatively prime to n/t , and when n is not square-free, all the summation is killed by $\mu(n) = 0$.

Finally, the factor $\sum_{t|n} O^{2g}(t)$ is nothing but the total number of elements in the $2g$ torus $(\mathbb{C}^\times)^{2g}$ killed by n , namely n^{2g} (see Remark 2.1.11), concluding

$$N_n^g(1) = \mu(n) n^{2g-3} n^{2g} = \mu(n) n^{4g-3}. \tag{5.31}$$

□

Another interesting fact about $E(q; \mathcal{M}^g(\mathrm{Sl}_n))$ deduced from formula (5.7) is the following

Corollary 5.8.2. *The E -polynomials $E_n(q) := E(q; \mathcal{M}^g(\mathrm{Sl}_n))$ of the Character Varieties $\mathcal{M}^g(\mathrm{Sl}_n)$ are palindromic. Thanks to Remark 5.2.4 this is equivalent to*

$$E_n(q) = E_n(q^{-1})q^{2(n^2-1)(g-1)}. \quad (5.32)$$

Proof. This follows from the identity 2.5 we recall here:

$$\mathcal{H}_\lambda(q) = (-1)^{|\lambda|} \mathcal{H}_{\lambda'}(q^{-1}).$$

Since, the exponent $2(g-1)$ kills every power of -1 and we end up with

$$\begin{aligned} E_n(q^{-1})q^{2(n^2-1)(g-1)} &= \sum_{\tau, t} q^{2(n^2-1)(g-1)} \left(q^{-\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q^{-1})}{q^{-1} - 1} \right)^{2(g-1)} t^{2g-1} C_\tau^t \\ &= \sum_{\tau, t} \left(q^{n^2-1} q^{-\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q^{-1})}{q^{-1} - 1} \right)^{2(g-1)} t^{2g-1} C_\tau^t \\ &= \sum_{\tau, t} \left(q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{1 - q} \right)^{2(g-1)} t^{2g-1} C_\tau^t \\ &= \sum_{\tau, t} \left(q^{\frac{n^2}{2}} \frac{\mathcal{H}_{\tau'}(q)}{q - 1} \right)^{2(g-1)} t^{2g-1} C_{\tau'}^t \\ &= E_n(q). \end{aligned} \quad (5.33)$$

since C_τ^t and $C_{\tau'}^t$ agree by Remark 5.2.5. □

Chapter 6

The Kitchen

In the next three sections we are going to prove the remaining claims from Chapter 5.

Lemma 6.1.1 proves the assertion of Remark 5.5.2 about (5.14).

Lemmas 6.2.1 and 6.2.2 take care of the statement made on Remark 5.4.4 and the one made right after equation (5.12), respectively. Lemma 6.2.3 proves the assertion made in Remark 5.5.6.

Finally, in Section 6.3 we prove Lemma 5.6.1, finishing with the calculation of the \widehat{Z} 's and therefore with the proof of Theorem 1.0.1.

6.1 The Sign

Here we see how many values the exponential factor in (5.14) can take, and find sufficient conditions to get rid of it.

Lemma 6.1.1. *The factor $\delta(\zeta_n)^{ds\binom{\hat{t}}{2}\sum|\Lambda(\gamma)|}$ from (5.13) is ± 1 . If q satisfies (1.9), then it is precisely 1.*

Proof. The exponent $ds\binom{\widehat{t}}{2}\sum|\Lambda(\gamma)| = ds\binom{\widehat{t}}{2}\sum\lambda_i$ could be written as

$$\begin{aligned} ds\binom{\widehat{t}}{2}\sum\lambda_i &= \frac{1}{2}s\widehat{d}\widehat{t}(\widehat{t}-1)\sum\lambda_i \\ &= \frac{1}{2}st\widehat{d}(\widehat{t}-1)\sum\lambda_i \\ &= \frac{1}{2}(q-1)\widehat{d}(\widehat{t}-1)\sum\lambda_i \end{aligned}$$

which is one half of an integer divisible by n , since $n|q-1$.

So the square of $\delta(\zeta_n)^{ds\binom{\widehat{t}}{2}\sum|\Lambda(\gamma)|}$ is $\delta(\zeta_n)^{(q-1)\widehat{d}(\widehat{t}-1)\sum\lambda_i}$ which is 1, leading us to the first assertion of this Lemma, namely

$$\delta(\zeta_n)^{ds\binom{\widehat{t}}{2}\sum|\Lambda(\gamma)|} = \pm 1.$$

For the second affirmation, we assume (1.9) and analyzing the exponent

$$\frac{1}{2}(q-1)\widehat{d}(\widehat{t}-1)\sum\lambda_i$$

we see it is already divisible by n .

If $2n$ divides $q-1$, then n divides $q-1/2$ and hence the whole exponent results divisible by n .

And when n is odd, the exponent is an integer number whose double is divisible by n , implying that n divides the exponent, already. \square

Remark 6.1.1. We could also point out that since $\widehat{t}|t$ and $t|n$, $\widehat{t}-1$ must be even, so there is the extra factor of 2 we were looking for.

6.2 The Vanishing

Let us now state and prove the vanishing lemmas. The proofs given here are rather elementary and follow from the same trick.

Lemma 6.2.1. $\widehat{S}(t, \tau) = 0$ if the type τ is not pure.

Proof. The idea here is to twist all the lowest degree characters multiplying them by a fixed factor of $\alpha \in \Gamma_1$, and get on one hand the original sum, but on the other one the same sum multiplied by a nontrivial factor, concluding thus that the whole sum is 0.

More precisely, given the type $\tau = \{m_{d,\lambda}\}$, d_1 the lowest degree appearing in the support of τ (i.e.: the minimum d such that $m_{d,\lambda} \neq 0$ for a nonempty $\lambda \in \mathcal{P}$), and character $\chi = \chi_\Lambda \in \text{Irr}(G)$ of type τ fixed by the action of $\delta^s \in \Gamma_1$ we consider a new character $\chi^\alpha \in \text{Irr}(G)$ corresponding to the function $\Lambda^\alpha \mathcal{P}_n^\Theta$ that maps every character in Γ to the same partition that Λ does, except for those characters $\gamma \in \Gamma_{d_1}$ that are going to be mapped to $\Lambda(\gamma\alpha^{-1})$. It is immediate that χ^α is also of type τ , fixed by the action of $\delta^s \in \Gamma_1$ and that $(\chi^\alpha)^{\alpha^{-1}} = \chi$, so $\chi \mapsto \chi^\alpha$ is actually a bijection and hence the twisted summation

$$\sum_{\substack{\chi \in \text{Irr}(G) \\ \delta^s \chi = \chi \\ \tau(\chi) = \tau}} \Delta_{\chi^\alpha}(\zeta_n)$$

is nothing but a permutation of the terms involved in (5.9), the sum for $\widehat{S}(t, \tau)$, and therefore leads to the same value.

But

$$\begin{aligned}
\Delta_{\chi^\alpha}(\zeta_n) &= \prod_{\{\gamma\}, \gamma \in \Gamma} \gamma^{|\Lambda^\alpha(\gamma)|}(\zeta_n) \\
&= \prod_{\{\gamma\}, \gamma \in \Gamma_{d_1}} \gamma^{|\Lambda^\alpha(\gamma)|}(\zeta_n) \prod_{\{\gamma\}, \gamma \notin \Gamma_{d_1}} \gamma^{|\Lambda^\alpha(\gamma)|}(\zeta_n) \\
&= \prod_{\{\gamma\}, \gamma \in \Gamma_{d_1}} \gamma^{|\Lambda(\gamma\alpha^{-1})|}(\zeta_n) \prod_{\{\gamma\}, \gamma \notin \Gamma_{d_1}} \gamma^{|\Lambda(\gamma)|}(\zeta_n) \\
&= \prod_{\{\gamma\}, \gamma \in \Gamma_{d_1}} (\gamma\alpha)^{|\Lambda(\gamma)|}(\zeta_n) \prod_{\{\gamma\}, \gamma \notin \Gamma_{d_1}} \gamma^{|\Lambda(\gamma)|}(\zeta_n)
\end{aligned} \tag{6.1}$$

which, factoring the α 's out becomes

$$\Delta_{\chi^\alpha}(\zeta_n) = \alpha(N_1^{d_1}(\zeta_n))^{\sum' |\Lambda(\gamma)|} \prod_{\{\gamma\}, \gamma \in \Gamma} \gamma^{|\Lambda(\gamma)|}(\zeta_n) \tag{6.2}$$

$$= \Delta_\chi(\zeta_n) \alpha(\zeta_n)^{\sum' |\Lambda(\gamma)| d_1} \tag{6.3}$$

$$= \Delta_\chi(\zeta_n) \alpha(\zeta_n)^{\sum' |\lambda| d_1 m_{d_1, \lambda}} \tag{6.4}$$

where the summations in the exponents of (6.2), (6.3) and (6.4) correspond respectively to:

- The sum of $|\Lambda(\gamma)|$ for every orbit $\{\gamma\}$ with γ of degree d_1
- The sum of $|\Lambda(\gamma)| d_1$ for every orbit $\{\gamma\}$ with γ of degree d_1
- The sum of $|\lambda| d_1 m_{d_1, \lambda}$ for every partition $\lambda \in \mathcal{P}$.

Going back to the sum (5.9) defining $\widehat{S}(t, \tau)$ we conclude:

$$\begin{aligned}
\widehat{S}(t, \tau) &= \sum_{\substack{\chi \in \text{Irr}(G) \\ \delta^s \chi = \chi \\ \tau(\chi) = \tau}} \Delta_{\chi^\alpha}(\zeta_n) \\
&= \sum_{\substack{\chi \in \text{Irr}(G) \\ \delta^s \chi = \chi \\ \tau(\chi) = \tau}} \Delta_{\chi}(\zeta_n) \alpha(\zeta_n)^{\sum |\lambda| d_1 m_{d_1, \lambda}} \\
&= \widehat{S}(t, \tau) \alpha(\zeta_n)^{\sum |\lambda| d_1 m_{d_1, \lambda}}.
\end{aligned} \tag{6.5}$$

If the exponent $\sum |\lambda| d_1 m_{d_1, \lambda}$ is not n , the twisting factor $\alpha(\zeta_n)^{\sum |\lambda| d_1 m_{d_1, \lambda}}$ could be nontrivial, since n is the order of ζ_n . For instance, taking $\alpha = \delta$ (or any other generator of Γ_1) does the job.

So $\widehat{S}(t, \tau) = 0$ unless the exponent $\sum_{\lambda \in \mathcal{P}} |\lambda| d_1 m_{d_1, \lambda}$ is n . But n is already the whole sum $\sum_{d, \lambda} |\lambda| d m_{d, \lambda}$. Thus the only way to have a nonzero $\widehat{S}(t, \tau)$ is for τ to be pure of degree δ_1 , that is to say, all the $\gamma \in \Gamma$ involved should have the same degree $d = d_1$. \square

Lemma 6.2.2. $\widehat{S}(t, \tau) = \sum_{\widehat{d}|d} S(t, \widehat{d}, \tau)$, that is to say, we could disregard all those terms $\Delta_{\chi}(\zeta_n)$ corresponding to characters χ with mixed newdegrees.

Proof. A similar trick works here. Assuming τ is a type pure of degree d , we define the new-type $\widehat{\tau}$ of a type τ character $\chi_{\Lambda} = \delta^s \chi_{\Lambda}$ and the map $\Lambda \in \mathcal{P}_n^{\Theta}$ associated to it, as the collection of new-multiplicities $\widehat{m}_{\widehat{d}, \lambda}$ given by the number of Frobenius orbits $\{[\gamma]\}$ for $[\gamma] \in \Gamma / \langle \delta^s \rangle$, of newdegree \widehat{d} that are mapped to $\lambda \in \mathcal{P}$ by Λ . Let us write $\widehat{\tau} \subseteq \tau$.

Then

$$\widehat{S}(t, \tau) = \sum_{\widehat{\tau} \subseteq \tau} S(t, \widehat{\tau}, \tau) \quad (6.6)$$

where $S(t, \widehat{\tau}, \tau)$ is the sum of $\Delta_\chi(\zeta_n)$ for those χ fixed by the action of δ^s , of type τ and new-type $\widehat{\tau}$.

Our claim will follow once we prove $S(t, \widehat{\tau}, \tau)$ vanishes unless $\widehat{\tau}$ is pure of some newdegree \widehat{d} , and noting that $S(t, \widehat{d}, \tau)$ is the sum of all such $S(t, \widehat{\tau}, \tau)$.

For the sums $S(t, \widehat{\tau}, \tau)$ we proceed in a similar way as before, twisting by a factor of $\alpha \in \Gamma_1$ all those γ of lowest new-degree \widehat{d}_1 . Since multiplication by α preserves both degrees and newdegrees it is easy to see that this twisting preserves both type and new-type.

In this case, from (6.6) we can pull out the factor

$$\alpha(\zeta_n)^{\sum |\lambda| t \widehat{d}_1 \widehat{m}_{\widehat{d}_1, \lambda}} \quad (6.7)$$

which could be made nontrivial for a suitable choice of α ($\alpha = \delta$ works again) unless the exponent $\sum |\lambda| t \widehat{d}_1 \widehat{m}_{\widehat{d}_1, \lambda}$ is n , and this can only happen when all the γ in the supports of the Λ 's have the same newdegree \widehat{d}_1 , in other words, $\widehat{\tau}$ is pure of newdegree \widehat{d}_1 . \square

The claim from Remark 5.5.6 will follow from next lemma.

Lemma 6.2.3. *Let n be a positive integer. Assume \widehat{d} , t and a list $\vec{\lambda} = (\lambda_1, \dots, \lambda_m)$ of positive integers are given such that $n = \widehat{d}t \sum \lambda_i$. Let q be a fixed prime power congruent to 1 modulo n , and note $s = (q-1)/t$. Let D be a*

multiple of \widehat{d} dividing \widehat{dt} . Write \widehat{t} for \widehat{dt}/D . Let Γ_D and Γ be as in Definition 2.2.4. Consider \vec{d} a list of degrees (d_1, \dots, d_m) all of which are multiples of \widehat{d} and divide D .

Then the sum

$$Z(\vec{d}, t, \widehat{d}, \vec{\lambda}) := \sum_{\substack{([\alpha_1], \dots, [\alpha_m]) \in (\Gamma_D / \langle \delta^s \rangle)^m \\ \{[\alpha_i]\} \neq \{[\alpha_j]\}, i \neq j \\ \deg(\alpha_i) = d_i \\ \deg([\alpha_i]) = \widehat{d}}} \prod_{i=1}^m \alpha_i^{\lambda_i}(\zeta_n^{\widehat{t}}) \quad (6.8)$$

will vanish if not all the entries from \vec{d} are the same (the evaluations in the product being computed as in Γ_D).

Proof. Let S be the subset of $(\Gamma_D / \langle \delta^s \rangle)^m$ consisting on all the m -tuples of character classes $([\alpha_1], \dots, [\alpha_m])$ as in the summation, namely $[\alpha_i] \in \Gamma_D / \langle \delta^s \rangle$ has degree \widehat{d} and α_i has degree d_i , no two of the entries $[\alpha_i]$ being $Frob_q$ -conjugates.

Since every character in Γ_1 has degree 1, multiplication by $\beta \in \Gamma_1$ changes neither degree nor newdegree. Then, we can define a free $\Gamma_1 / \langle \delta^s \rangle$ action on S as follows. A class character $[\beta] \in \Gamma_1 / \langle \delta^s \rangle$ acts on the m -tuple $([\alpha_1], \dots, [\alpha_m])$ by coordinatewise multiplication on those entries $[\alpha_i]$ correspond to those i with $d_i = \min \{d_j\}$ (it is easy to see that in the new m -tuple no two of the entries are $Frob_q$ -conjugates).

We could regard $(\Gamma_D / \langle \delta^s \rangle)^m$ as the dual of an m -fold product of copies

of $\text{Ker}(^T N_1^D(\delta^s))^{-1}$, and $\Gamma_{1/\langle \delta^s \rangle}$ as a the subgroup of $\left(\Gamma_{D/\langle \delta^s \rangle}\right)^m$ of m -tuples $([\beta_1], \dots, [\beta_m])$ with trivial entries on those i with $d_i > \min \{d_j\}$ and the same character class $[\beta]$ in all others.

The result will follow by direct application of Lemma 1.0.5 from Appendix, once we prove that the element $(\zeta_n^{\hat{t}}, \dots, \zeta_n^{\hat{t}}) \in \text{Ker}(^T N_1^D(\delta^s))^m$ at which we are evaluating all the summation does not belong to the intersection of kernels from the statement.

And this is indeed true since, regarding $[\beta]$ as an m -tuple, evaluation at $\zeta_n^{\hat{t}}$ becomes

$$[\beta](\zeta_n^{\hat{t}}) = \beta(\zeta_n^{\hat{t}})^{D \sum' \lambda_i} = \beta(\zeta_n)^{\hat{t} D \sum' \lambda_i}$$

where the sum $\sum' \lambda_i$ from the exponents is over those λ_i with $d_i = \min \{d_j\}$. Taking $\beta = \delta$ and keeping in mind that ζ_n is of order n and that not all of the d_i are the same we get a nontrivial value for $[\beta](\zeta_n^{\hat{t}})$. \square

6.3 The Values

This whole section is devoted to the proof of Lemma 5.6.1, namely that the sums $\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda})$ defined in (5.18) satisfy

$$\widehat{Z}(D, t, \widehat{d}, \overrightarrow{\lambda}) = \begin{cases} \mu(\widehat{d})(-\widehat{d})^{m-1}(m-1)! \frac{q-1}{t} & \text{when } \gcd(D, t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

¹i.e.: the subgroup of $\mathbb{F}_{q^D}^\times$ whose dual is $\Gamma_{D/\langle \delta^s \rangle}$ by Remark 2.2.5.

Let us remind the definition of $\widehat{Z}(D, t, \widehat{d}, \vec{\lambda})$

$$\widehat{Z}(D, t, \widehat{d}, \vec{\lambda}) = \sum_{\substack{([\alpha_1], \dots, [\alpha_m]) \in (\Gamma^D / \langle \delta^s \rangle)^m \\ \{[\alpha_i]\} \neq \{[\alpha_j]\}, i \neq j \\ \deg([\alpha_i]) = \widehat{d}}} \prod_{i=1}^m \alpha_i^{\lambda_i}(\zeta_n^{\widehat{t}})$$

where we are assuming

- q is a fixed prime power satisfying (1.9),
- $D, \widehat{d}, t, \widehat{t}$ and n are positive integers,
- \widehat{d} is a divisor of D ,
- $\widehat{d}t = D\widehat{t}$ and
- $\vec{\lambda} = (\lambda_1, \dots, \lambda_m)$ an m -tuple of positive integers satisfying

$$n = D\widehat{t} \sum_{i=1}^m \lambda_i, \quad (6.9)$$

and the conventions are that s stands for $(q-1)/t$, δ is some fixed generator of $\Gamma_1 = \widehat{\mathbb{F}}_q^\times$, and $\zeta_n \in \mathbb{F}_q^\times$ an element of order n .

Remark 6.3.1. The sum is taken over all possible m -tuples $([\alpha_1], \dots, [\alpha_m])$ of classes of characters $[\alpha_i] \in \Gamma^D / \langle \delta^s \rangle$ of newdegree exactly \widehat{d} with the additional condition that no two of the $[\alpha_i]$ can be conjugate by the action of Frobenius, in particular, they must be different. In other words, the orbits $\{[\alpha_i]\}$ and $\{[\alpha_j]\}$ must be disjoint, and since they are orbits, it is the same to say that they are different.

Remark 6.3.2. The α_i are evaluated as in Γ_D .

The outline of the proof goes as follows:

first: realize \widehat{Z} as one value of a particular function f on certain poset Π^ρ of set-partitions fixed by an action.

second: define the accumulated sums \widehat{f} of f .

third: compute \widehat{f} .

fourth: solve for f using Möbius Inversion Formula for \widehat{f} on Π^ρ .

6.3.1 First step: The poset Π^ρ .

Let us consider the poset $\Pi^\rho = \Pi_{m\widehat{d}}^\rho$ (as in example 2.1.9 on 2.1.4) of set-partitions of $\{1, 2, \dots, m\widehat{d}\}$ fixed by the permutation ρ consisting of m disjoint \widehat{d} -cycles

$$\rho = (1, \dots, \widehat{d})(\widehat{d} + 1, \dots, 2\widehat{d}) \dots ((m-1)\widehat{d} + 1, \dots, m\widehat{d}) \in \mathbb{S}_{m\widehat{d}}. \quad (6.10)$$

To every such set-partition ν assign $f(\nu)$ defined as the following sum:

The range of the summation will be the set $R(\nu)$ of any possible ordered $m\widehat{d}$ -tuple $\vec{\alpha} = ([\alpha_j])_{j=1}^{m\widehat{d}}$ of classes of characters in $\Gamma_D / \langle \delta^s \rangle$ satisfying both

- 1 $[\alpha_{\rho(i)}] = [\alpha_i]^q$, so ρ acts as Frobenius.
- 2 $[\alpha_i] = [\alpha_j]$ if and only if i and j belong to the same part of ν , so the set-partition ν defines all the equality relations among the classes of characters in the $m\widehat{d}$ -tuple.

And the number we sum for $\vec{\alpha} \in R(\nu)$ is computed as

$$\prod_{\substack{i,j \\ 1 \leq i \leq m \\ j=(i-1)\widehat{d}+1}} \alpha_j^{\lambda_i}(\zeta_n^{\widehat{t}}) \quad (6.11)$$

where each evaluation is made in Γ_D .

Remark 6.3.3. A couple of things should be checked for this definition to make sense.

The evaluation $\alpha(\zeta_n^{\widehat{t}})$ in Γ_D is well defined, that is to say, it does not depend on the representative α chosen from the class $[\alpha] \in \Gamma_D / \langle \delta^s \rangle$.

To see this we must check that both $\alpha(\zeta_n^{\widehat{t}})$ and $(\delta^s \alpha)(\zeta_n^{\widehat{t}})$ have the same value when the evaluation is done in Γ_D . According to Remark 2.2.5 this is the same that $\zeta_n^{\widehat{t}} \in \text{Ker}(^T N_1^D(\delta^s))$.

This is indeed true since:

$$(\delta^s \alpha)(\zeta_n^{\widehat{t}}) = \delta^s(N_1^D(\zeta_n^{\widehat{t}}))\alpha(\zeta_n^{\widehat{t}}) = \delta(\zeta_n^{sD\widehat{t}})\alpha(\zeta_n^{\widehat{t}}) = \alpha(\zeta_n^{\widehat{t}})$$

because $\zeta_n \in \mathbb{F}_q$, hence $N_D(\zeta_n) = \zeta_n^D$, and since

$$sD\widehat{t} = st\widehat{d} = (q-1)\widehat{d}$$

which is divisible by n , then $\delta(\zeta_n^{sD\widehat{t}}) = \delta(1) = 1$.

With this setting, $\widehat{Z}(D, t, \widehat{d}, \vec{\lambda})$ becomes:

$$f(\widehat{0}) = \sum_{\vec{\alpha} \in R(\widehat{0})} \prod_{\substack{i,j \\ 1 \leq i \leq m \\ j=(i-1)\widehat{d}+1}} \alpha_j^{\lambda_i}(\zeta_n^{\widehat{t}})$$

for $\widehat{0}$ the finest set partition (i.e. that with $m\widehat{d}$ parts of one element each).

6.3.2 Second step: The accumulated sums \widehat{f}

The good thing about this setting is that it is fairly easy to compute the accumulated sums for f defined as

$$\widehat{f}(\nu) := \sum_{\varrho \geq \nu} f(\varrho) \quad (6.12)$$

where the restrictions given by the set partition ϱ are not necessarily strict, in other words, there could be repetitions among the classes of characters corresponding to different parts of ϱ .

More precisely

$$\widehat{f}(\nu) = \sum_{\vec{\alpha} \in \widehat{R}(\nu)} \prod_{\substack{i,j \\ 1 \leq i \leq m \\ j=(i-1)\widehat{d}+1}} \alpha_j^{\lambda_i}(\zeta_n^t) \quad (6.13)$$

where

$$\widehat{R}(\nu) := \bigcup_{\varrho \geq \nu} R(\varrho) \quad (6.14)$$

is defined analogously to $R(\nu)$ with the second condition in (6.3.1) relaxed to:

2' $[\alpha_i] = [\alpha_j]$ if i and j belong to the same part of ν .

6.3.3 Third Step: Calculating \widehat{f}

Our next task is to prove that most of the $\widehat{f}(\nu)$ are zero except, possibly, for the top one, the one corresponding to the partition $\widehat{1}$ having only one part.

In 6.3.3.1 we prove this by pulling out a factor in the sum (6.13). Such factor resulting a full character sum of a cyclic group. To prove that the

element at which the characters are being evaluated is nontrivial we count the size of certain torsion subgroup by computing gcd's.

The calculation of $\widehat{f}(\widehat{1})$ is finished in 6.3.3.2 when the size of the aforementioned cyclic group is finally computed.

6.3.3.1 The Killing Factor

To compute $\widehat{f}(\nu)$ for a particular $\nu \in \Pi^\rho$, let us consider ν_1 the part of ν containing the index 1.

Let us define ν_s by

$$\nu_s := \bigcup_i \rho^i(\nu_1)$$

the union of ρ -conjugates from ν_1 .

Since the conditions on the classes of characters $[\alpha_i]$ makes them to coincide when their indices belong to the same part of ν , but there is no condition making them necessarily different when their indices belong to different parts, we can pull out a factor of

$$\widehat{f}_1(\nu) := \sum_{\widehat{R}_1(\nu)} \prod_{\substack{i,j \\ 1 \leq i \leq m \\ j=(i-1)d+1 \\ j \in \nu_s}} \alpha_j^{\lambda_i}(\zeta_n^t) \quad (6.15)$$

from $\widehat{f}(\nu)$, where the range of summation $\widehat{R}_1(\nu)$ consists in all possible tuples of classes of characters $[\alpha_i]$, with the set of indices restricted to the union of ρ -orbits of elements in ν_1 with the same conditions:

- $[\alpha_{\rho(i)}] = [\alpha_i]^q$, so ρ acts as Frobenius.

- $[\alpha_i] = [\alpha_j]$ if i and j belong to the same part of ν .

Our next task is to prove that such factor $\widehat{f}_1(\nu)$ will vanish (making the whole $\widehat{f}(\nu)$ vanish as well) unless, possibly, when ν_1 is the whole partition ν . Equivalently, for $\widehat{f}(\nu)$ to be nonzero we must have $\nu = \widehat{1}$ the partition with only one part.

Remark 6.3.4. We must stress that since $\zeta_n \in \mathbb{F}_q$, the product in the summation (6.15) defining $\widehat{f}_1(\nu)$ could be regarded as a product of repeated classes of characters. This is because

$$\alpha^q(\zeta_n^{\widehat{t}}) = \alpha(\zeta_n^{q\widehat{t}}) = \alpha(\zeta_n^{\widehat{t}})$$

and by definition of ν_1 the α_i in the product are $Frob_q$ -conjugates.

Hence

$$\prod_{\substack{i,j \\ 1 \leq i \leq m \\ j=(i-1)\widehat{d}+1 \\ j \in \nu_s}} \alpha_j^{\lambda_i}(\zeta_n^{\widehat{t}}) = \alpha_1^{\sum' \lambda_i}(\zeta_n^{\widehat{t}}) \quad (6.16)$$

where the exponent $\sum' \lambda_i$ is the sum of those λ_i with $1 \leq i \leq m$ such that $(i-1)\widehat{d}+1 \in \nu_s$.

Remark 6.3.5. The presence of a subindex $k \in \nu_1$ greater than 1 but smaller than $\widehat{d}+1$ makes the class $[\alpha_1]$ to have a Frobenius orbit of a smaller size than \widehat{d} , that is to say $[\alpha_1]^{q^{k-1}} = [\alpha_k] = [\alpha_1]$.

Thus, the factor $\widehat{f}_1(\nu)$ end up being:

$$\widehat{f}_1(\nu) = \sum_{\widehat{R}_1(\nu)} \alpha_1(\zeta_n^{\widehat{t} \sum' \lambda_i}) \quad (6.17)$$

where the $[\alpha_1]$ ranges over the subgroup of $\Gamma_{D/\langle\delta^s\rangle}$ of classes of characters fixed by $Frob_q^{\widehat{d}_1}$, where \widehat{d}_1 is the number of parts ν_j of ν containing a ρ -conjugate of 1.²

Remark 6.3.6. If $\widehat{d}_1 = \widehat{d}$ then all the $i \in \nu_1$ are congruent to 1 modulo \widehat{d} , and the character class $[\alpha_1]$ ranges over all $[\alpha] \in \Gamma_{D/\langle\delta^s\rangle}$ satisfying $[\alpha]^{q^{\widehat{d}}} = [\alpha]$.

Remark 6.3.7. If $\widehat{d}_1 = 1$, we must have $\{1, 2, \dots, \widehat{d}\} \subseteq \nu_1$ and $[\alpha_1]$ moves on the subgroup of newdegree 1 character classes of $\Gamma_{D/\langle\delta^s\rangle}$, namely those with $[\alpha]^q = [\alpha]$.

Remark 6.3.8. In (6.17) we are computing a character sum (as in Remark 2.2.8) over

$$\left(\Gamma_{D/\langle\delta^s\rangle}\right)_{\widehat{d}_1} := \left\{ [\alpha] \in \left(\Gamma_{D/\langle\delta^s\rangle}\right) \mid [\alpha] = [\alpha]^{q^{\widehat{d}_1}} \right\} \quad (6.18)$$

a subgroup of a quotient of the group of characters Γ_D , and we are evaluating them at $\zeta_n^{\widehat{t}\sum' \lambda_i}$, with the summation in the exponent as explained in (6.16).

Remark 6.3.9. Since the character $\delta \in \Gamma_1$ has degree 1, it must also have newdegree 1 (by 5.4.3), and hence it belongs to $\left(\Gamma_{D/\langle\delta^s\rangle}\right)_{\widehat{d}_1}$.

Remark 6.3.10. The fact that we are considering the quotient $\Gamma_{D/\langle\delta^s\rangle}$ amounts of taking the character group of the subgroup of $\mathbb{F}_{q^D}^\times$ killed by δ^s (by Remark 2.2.5), namely

$$\Gamma_{D/\langle\delta^s\rangle} = \text{Hom} \left(\text{Ker} \left({}^T N_1^D(\delta^s) \right), \mathbb{C}^\times \right), \quad (6.19)$$

and we have already checked that $\zeta_n^{\widehat{t}}$ belongs to such kernel.

²in other words, \widehat{d}_1 is the degree of ν_1 under the induced ρ -action.

Remark 6.3.11. The fact that the sum is taken over a subgroup of a character group corresponds to take a full character sum on a quotient of such group (also by Remark 2.2.5), which amounts to identify those elements in $\text{Ker} \left({}^T N_1^D(\delta^s) \right)$ whose quotient belongs to the intersection K of the kernels of the $[\alpha]$'s involved in the sum (6.17). In other words

$$K := \bigcap_{[\alpha] \in (\Gamma_D / \langle \delta^s \rangle)_{\hat{d}_1}} \text{Ker}([\alpha]) \subseteq \text{Ker} \left({}^T N_1^D(\delta^s) \right) \subseteq \mathbb{F}_{q^D}^\times \quad (6.20)$$

and

$$(\Gamma_D / \langle \delta^s \rangle)_{\hat{d}_1} = \text{Hom} \left(\text{Ker} \left({}^T N_1^D(\delta^s) \right) /_K, \mathbb{C}^\times \right). \quad (6.21)$$

By Remarks 6.3.8, 6.3.10 and 6.3.11 we conclude the following

Lemma 6.3.1. *The character sum (6.17) will be zero if $\zeta_n^{\hat{\Sigma}' \lambda_i}$ does not belong to the intersection of kernels K , And if it does, the result will be the order of the group $(\Gamma_D / \langle \delta^s \rangle)_{\hat{d}_1}$.*

Since Γ_D is cyclic, then so is $\Gamma_D / \langle \delta^s \rangle$ and its subgroup $(\Gamma_D / \langle \delta^s \rangle)_{\hat{d}_1}$.

The order of $\Gamma_D / \langle \delta^s \rangle$ is

$$\frac{|\Gamma_D|}{|\langle \delta^s \rangle|} = \frac{q^D - 1}{t}$$

and in a cyclic group of such order, the subgroup of elements fixed by $\text{Frob}_q^{\hat{d}_1}$ (which by definition is that of those elements killed by $q^{\hat{d}_1} - 1$) agrees with the one of elements killed by

$$\gcd \left(\frac{q^D - 1}{t}, q^{\hat{d}_1} - 1 \right) \quad (6.22)$$

and thus this must also be its order.

Lemma 6.3.2. *The order of $\left(\Gamma^D/\langle\delta^s\rangle\right)_{\widehat{d}_1}$ is*

$$\frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\frac{D}{\widehat{d}_1}, t\right).$$

Proof. Keeping in mind that $t|n$ and $n|q-1$ we could simplify (6.22) to

$$\begin{aligned} \gcd\left(\frac{q^D - 1}{t}, q^{\widehat{d}_1} - 1\right) &= \frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\underbrace{\left(q^{\widehat{d}_1}\right)^{\frac{D}{\widehat{d}_1} - 1} + \dots + q^{\widehat{d}_1} + 1}_{D/\widehat{d}_1 \text{ terms}}, t\right) \\ &= \frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\underbrace{1 + \dots + 1 + 1}_{D/\widehat{d}_1 \text{ terms}}, t\right) \\ &= \frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\frac{D}{\widehat{d}_1}, t\right). \end{aligned} \tag{6.23}$$

The Lemma is thus proved. \square

Remark 6.3.12. Plugging this result in Lemma 6.3.1, we conclude that the value of the character sum (6.17)

$$\widehat{f}_1(\nu) = \sum_{\widehat{R}_1(\nu)} \alpha_1(\zeta_n^{\widehat{t}\Sigma' \lambda_i}) \tag{6.24}$$

will be either

$$\frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\frac{D}{\widehat{d}_1}, t\right)$$

or 0 depending on whether $\zeta_n^{\widehat{t}\Sigma' \lambda_i}$ belongs or not to the intersection K from (6.20).

We are now in conditions to prove the following

Lemma 6.3.3. *All the accumulated sums $\widehat{f}(\nu)$ vanish except, possibly, for the top one $\widehat{f}(\widehat{1})$.*

Proof. Let us assume $\widehat{f}(\nu) \neq 0$.

Since we factored out $\widehat{f}_1(\nu)$ from the sum (6.13), the vanishing of former implies that of the latter.

From Lemma 6.3.1 we know $\widehat{f}_1(\nu)$ will be zero unless $\zeta_n^{\widehat{t}\sum' \lambda_i} \in K$.

K was defined as an intersection of kernels in (6.20), and one of the character classes involved in such intersection is $[\delta]$ (Remark 6.3.9).

Let us see that $\zeta_n^{\widehat{t}\sum' \lambda_i}$ does not belong to $\text{Ker}([\delta])$ unless the sum $\sum' \lambda_i$ in the exponent (as defined in (6.16)) is the whole sum $\sum \lambda_i = \frac{n}{D\widehat{t}}$.

Evaluation of $\delta \in \Gamma_D$ at $\zeta_n^{\widehat{t}\sum' \lambda_i}$ gives:

$$\delta(N_1^D(\zeta_n^{\widehat{t}\sum' \lambda_i})) = \delta(\zeta_n^{D\widehat{t}\sum' \lambda_i})$$

(evaluated as in Γ_1) and for this to be 1, we must have $n|D\widehat{t}\sum' \lambda_i$ because ζ_n is of order n and δ generates $\widehat{\mathbb{F}}_q^\times$.

Since $n = D\widehat{t}\sum \lambda_i$ (see assumption (6.9)) we conclude

$$\sum' \lambda_i = \sum \lambda_i = \frac{n}{D\widehat{t}},$$

which means all the exponents λ_i are involved in the sum \sum' .

According to (6.16) and the definition of $\widehat{R}_1(\nu)$ (right after (6.15)), in order to have $\zeta_n^{\widehat{t}\sum' \lambda_i}$ killed by δ (viewed in Γ_D) we must have

$$\nu = \bigcup_i \rho^i(\nu_1) \tag{6.25}$$

and thus

$$\widehat{R}(\nu) = \widehat{R}_1(\nu) \quad (6.26)$$

therefore $\sum_{\widehat{R}_1(\nu)} \alpha_1^{\sum' \lambda_i}(\zeta_n^t)$ is not just a factor of (6.13), but the whole sum

$$\widehat{f}(\nu) = \sum_{\vec{\alpha} \in \widehat{R}(\nu)} \prod_{\substack{i,j \\ 1 \leq i \leq m \\ j = (i-1)\widehat{d}+1 \\ j \in \nu}} \alpha_j^{\lambda_i}(\zeta_n^t).$$

So far we know that $\widehat{f}_1(\nu)$ will vanish unless (6.25) holds, and in that case $\widehat{f}_1(\nu) = \widehat{f}(\nu)$. Let us see that it also vanishes unless $\nu = \nu_1$, that is to say, $\nu = \widehat{1}$.

Taking a closer look at

$$\zeta_n^{t \sum' \lambda_i} = \zeta_n^{t \sum \lambda_i} = \zeta_n^{\frac{n}{D}} = \zeta_D$$

we see that the element at which we are evaluating the character classes in (6.17) is an order D element.

According to Lemma 6.3.1, in order to compute $\widehat{f}_1(\nu)$ we only need to check whether ζ_D belongs or not to the intersection K from (6.20), and since K is cyclic, this is equivalent as checking whether D divides or not the order of K (Remark 2.2.9).

Analyzing formula (6.21) from Remark 6.3.11 and Remark 6.3.10, we see that the order of K is the quotient of the orders of $\Gamma^D / \langle \delta^s \rangle$ and $\left(\Gamma^D / \langle \delta^s \rangle \right)_{\widehat{d}_1}$, which by Lemma 6.3.2 is the quotient

$$\frac{\frac{q^D - 1}{t}}{\frac{q^{\widehat{d}_1} - 1}{t} \gcd\left(\frac{D}{\widehat{d}_1}, t\right)} = \frac{q^D - 1}{(q^{\widehat{d}_1} - 1) \gcd\left(\frac{D}{\widehat{d}_1}, t\right)} \quad (6.27)$$

and an order D element like ζ_D will belong to it if and only if D divides its order (Remark 2.2.6).

For D to divide $\frac{q^D - 1}{(q^{\hat{d}_1} - 1) \gcd\left(\frac{D}{\hat{d}_1}, t\right)}$ it is necessary that it divides

$$\frac{q^D - 1}{q^{\hat{d}_1} - 1} = \left(q^{\hat{d}_1}\right)^{\frac{D}{\hat{d}_1} - 1} + \dots + q^{\hat{d}_1} + 1 \equiv \frac{D}{\hat{d}_1} \pmod{q - 1}$$

but $D|n$ and $n|q - 1$. So if D divides $\frac{D}{\hat{d}_1}$ that means $\hat{d}_1 = 1$.

By Remark 6.3.7, and (6.25) we conclude that the only way that D divides the order of K (and hence $\hat{f}(\nu)$ could be nonzero) is if

$$\nu = \bigcup_i \rho^i(\nu_1) = \nu_1$$

meaning ν is the top set-partition $\hat{1}$, the one with only one part. \square

6.3.3.2 The Leftovers

So far we know that the only potentially nonzero accumulated sum $\hat{f}(\nu)$ is the top one $\hat{f}(\hat{1}) = f(\hat{1})$, therefore $f(\hat{0}) = \mu(\hat{0}, \hat{1})\hat{f}(\hat{1})$ by Möbius Inversion Formula.

According to Remark 6.3.8, Lemma 6.3.1, and (6.27) from the proof of Lemma 6.3.3, $\hat{f}(\hat{1})$ could still be either $^3 \frac{q - 1}{t} \gcd(D, t)$ or zero, depending on whether D divides or not $\frac{q^D - 1}{(q - 1) \gcd(D, t)}$.

Let us further analyze the quotient $\frac{q^D - 1}{q - 1}$.

³the order of $\left(\Gamma_{D/\langle \delta^s \rangle}\right)_1$.

By the Newton binomial expansion

$$\begin{aligned}
q^D - 1 &= (q - 1 + 1)^D - 1 \\
&= -1 + 1 + (q - 1)D + (q - 1)^2 \binom{D}{2} + \sum_{k \geq 3} (q - 1)^k \binom{D}{k} \\
&= (q - 1)D + (q - 1)^2 \binom{D}{2} + O((q - 1)^3)
\end{aligned}$$

where $O((q - 1)^k)$ stands for an integer multiple of $(q - 1)^k$.

Then, to get a nonzero $\widehat{f}(\widehat{1})$, D must divide

$$\begin{aligned}
\frac{q^D - 1}{(q - 1) \gcd(D, t)} &= \frac{(q - 1)D + (q - 1)^2 \binom{D}{2} + O((q - 1)^3)}{(q - 1) \gcd(D, t)} \\
&= \frac{D + (q - 1) \binom{D}{2} + O((q - 1)^2)}{\gcd(D, t)}
\end{aligned}$$

or, what is the same, $\gcd(D, t)$ must divide

$$\frac{D + (q - 1) \binom{D}{2} + O((q - 1)^2)}{D} = 1 + \frac{(q - 1)(D - 1)}{2} + \frac{(q - 1)}{D} O(q - 1).$$

Keeping in mind that $\gcd(D, t) | D$ and $D | q - 1$ we can disregard the last term in the right hand side and end up with the condition $\gcd(D, t)$ divides $1 + \frac{(q - 1)(D - 1)}{2}$.

Now, for n odd, $D - 1$ results even and $\frac{(q - 1)(D - 1)}{2}$ is a multiple of D , hence of $\gcd(D, t)$ and we get $\gcd(D, t) = 1$.

For even n but $2n | q - 1$, the term $\frac{(q - 1)}{2}(D - 1)$ is again multiple of n and hence of D , therefore $\gcd(D, t)$ is again 1.

Summarizing, if q satisfies 1.9, $\widehat{f}(\widehat{1})$ will be zero unless $\gcd(D, t) = 1$, in which case it will be the order of the order of $\left(\Gamma_{D/\langle \delta^s \rangle} \right)_1$.

6.3.4 Fourth step: The Inversion

The assertion will now follow from an inversion formula on the poset Π^ρ , and the result of Hanlon on the Möbius function of this poset ([8] Theorem 4).

Under the current conditions, the only nonzero $\widehat{f}(\nu)$ is $\widehat{f}(\widehat{1}) = f(\widehat{1})$ and hence

$$f(\widehat{0}) = \mu(\widehat{0}, \widehat{1}) \widehat{f}(\widehat{1}) = \begin{cases} \mu(\widehat{0}, \widehat{1}) \frac{q-1}{t} \gcd(D, t) & \text{if } \gcd(D, t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

But

$$\mu(\widehat{0}, \widehat{1}) = \mu(\widehat{d})(-\widehat{d})^{m-1}(m-1)! \quad (6.28)$$

and the factor $\gcd(D, t)$ is trivially removed in the first case, so we finally proved Lemma 5.6.1:

$$Z(D, t, \widehat{d}, \vec{\lambda}) = f(\widehat{0}) = \begin{cases} \mu(\widehat{d})(-\widehat{d})^{m-1}(m-1)! \frac{q-1}{t} & \text{if } \gcd(D, t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Appendix

Appendix 1

Two Lemmas from Group Theory

In this appendix we will state and prove two lemmas we used during the computation of the number of points of $\mathcal{M}^g(\mathrm{Sl}_n(\mathbb{F}_q))$ that are group-theoretical in nature.

Lemma 1.0.4. *Let G and H be two finite groups acting on a set X . Let us make the further assumption that both actions commute, that is to say $g \cdot (h \cdot x) = h \cdot (g \cdot x)$ for all $g \in G$, $h \in H$ and $x \in X$. There is a natural G -action on the set of orbits X/H given by $g \cdot Hx = H(g \cdot x)$ ¹. Let us note $[x]$ the H -orbit of $x \in X$. Then for any $x \in X$, the size of the G -orbit of $[x]$ divides that of the G -orbit of x .*

Proof. The G -stabilizer of x is a subgroup of the G -stabilizer of $[x]$, hence the index of the latter divides that of the former. The lemma follows since the indexes coincide with the sizes of the corresponding G -orbits. \square

Lemma 1.0.5. *Let G be an abelian group, $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$ its dual, g an element of G , and $\widehat{Q} \subseteq \widehat{G}$ a subgroup. Pointwise multiplication defines an*

¹well defined since both actions commute

action $\widehat{Q} \curvearrowright \widehat{G}$, and consider S a subset of \widehat{G} stable by this \widehat{Q} -action, and the character sum:

$$S(g) := \sum_{\phi \in S} \phi(g). \quad (1.1)$$

If g does not belong to $K = \bigcap_{\psi \in \widehat{Q}} \text{Ker}(\psi)$ then $S(g) = 0$.

Proof. Since \widehat{Q} is a subgroup of \widehat{G} , it must be (by Remark 2.2.5) the dual of some quotient $Q = G/K$ of G , with K as in the statement (i.e.: the intersection of kernels of the characters in \widehat{Q}).

The fact that $g \notin K$ amounts for the class $[g] \in Q$ being nontrivial.

Then, the character sum

$$\widehat{Q}(g) := \sum_{\psi \in \widehat{Q}} \psi(g) \quad (1.2)$$

vanishes, since it is a full character sum of a nontrivial element.

The Lemma follows then from the observation that the condition on S being \widehat{Q} -stable makes S the disjoint union of \widehat{Q} -orbits, and each of them is a multiple of (1.2), which is zero under our hypothesis. \square

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Vita

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